MEASURE THEORY ON TIME SCALES

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ABSTRACT

MEASURE THEORY ON TIME SCALES

In this thesis, we have studied measure theory adapted to time scales. delta and nabla-measures were first defined by Guseinov in 2003, then in a further study, the relationship between Lebesgue delta-integral and Riemann delta-integral were introduced in detail by Guseinov and Bohner. In 2004, Cabada established the relationship between delta-measure and the classical Lebesgue measure, moreover, Lebesgue delta-integral and the classical Lebegue integral. Finally, delta-measurability of sets was studied by Rzezuchovsky in 2005. In this study, we have adapted basic concepts of the measure theory to time Scales, by using definitions and properties given in these papers. With the help of related papers, Lebesgue-Stieltjes measure has been constructed on time scales and the link between Lebesgue-Stieltjes measure and Lebesgue-Stieltjes delta-measure and also link between Lebesgue-Stieltjes delta-integral and Lebesgue-Stieltjes integral have taken place.

ÖZET

ZAMAN SKALASINDA ÖLÇÜ TEORİSİ

Bu tezde, zaman skalasına uyarlanmış ölçü teorisini çalıştık. Lebesgue delta ve nabla-ölçümü ilk olarak Guseinov tarafından 2003'te tanımlanmış, daha geniş bir çalışmayla Guseinov ve Bohner, Lebesgue delta-integral ve Riemann delta-integral arasındaki ilişkiyi ortaya koymuştur. 2004'te Cabada, delta-ölçümü ve klasik Lebesgue ölçümü, bunun yanında Lebesgue delta-integral ve klasik Lebesgue delta-integral arasındaki ikişkiyi açıklamıştır ve 2005'te Rzezuchovski kümelerin delta-ölçülebilirliği üzerine çalışmıştır. Bu çalışmada, belirtilen makalelerideki tanım ve özellikleri kullanarak ölçüm kuramındaki temel kavramları zaman skalasına uyarladık. Belirtilen makalelerin yardımıyla Lebesgue-Stieltjes delta-ölçümünü tanımladık ve Lebesgue-Stieltjes delta-integral ve Lebesgue-Stieltjes integral arasındaki bağlantıya yer verdik.

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CHAPTER 1

INTRODUCTION TO MEASURE AND TIME SCALES

Measure theory was born in order to generalize a geometric concept the ”length” to subsets of the real numbers , then area and volume applications took place by applying the theory to two and three dimensional spaces. In this thesis, the classical Lebesgue measure theory has been adapted to time scales and for simplicity, we have been concerned about the real line. The measure constructed on time scales is different from the classical Lebesgue measure. In the classical Lebesgue measure, single point set has measure zero, consequently for $a, b \in \mathbb{R}$ and $a \leq b$, measures of $[a, b]$, $[a, b)$, (a, b), $(a, b]$ are equal, that is, the difference of endpoints, whereas, for the measure constructed on a time scale, the single point set may have measure different from zero, depending on the character of the point, as a result, it is natural that different types of intervals with the same endpoints may have different measures.

Observing a special case, that is time scale is the real numbers itself, then two measures, Lebesgue measure and the measure on time scales coincide. Furthermore, for the case that the time scale is the integers, then any intervals measure turns into its cardinality number.

Finally, there is a strict relation between the classical Lebesgue integral and the integral constructed on measure on time scales and the difference between two structures comes from the difference of single point set’s measures.

1.1. Basic Concepts in Measure Theory

Around the beginning of the 20th century, the theory of measure was originated. At that time it was realized that, in order to get a better understanding of the structure of functions it was necessary to make a through study of the subsets of Euclidian spaces. To study these sets, it became clear that notions such as ”length” and ”area” needed to be generalized.

E. Borel in 1898 was the first to establish a measure theory on the subsets of the real numbers known as Borel sets. In 1902, H. Lebesgue presented Lebesgue measure and integral constructed on measure theory which could be applied to a wider class of functions than the Riemann integral did. In 1918, C. Caratheodory introduced and studied on outer measures. And especially in the first half of the 20th century, other developments followed on this theory.

In this section, we will introduce the general concept of measure, Caratheodory extension, measurable functions and integral of measurable functions.

Definition 1.1 *Let $\mathcal{A}$ be a nonempty collection of subsets of a nonempty set X. The collection $\mathcal{A}$ is called an algebra of sets (or simply an algebra) if it satisfies the following properties:*

i) *If $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$.*

ii) *If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$, $A^c = X \setminus A$.*

In addition to being an algebra, if $\bigcup\limits_{n=1}^{\infty} A_n$ belongs to $\mathcal{A}$ for every sequence $\{A_n\}_{n\in\mathbb{N}}$, $\mathcal{A}$ is said to be a σ-algebra and the pair $(X, \mathcal{A})$ is called a measurable space and the sets in $\mathcal{A}$, measurable sets.

Definition 1.2 *Let $(X, \mathcal{A})$ be a measurable space. A set function $\mu : \mathcal{A} \to [0, \infty]$ is said to be a measure if*

i) $\mu(\bigcup\limits_{j=1}^{\infty} A_j) = \sum\limits_{j=1}^{\infty} \mu(A_j)$, $A_j \in \mathcal{A}$, $j = 1, 2, ...$, *that is, μ is countably additive.*

ii) $\mu(\emptyset) = 0$.

If $\mathcal{A}$ is a σ-algebra on a set X and μ is countably additive measure on $\mathcal{A}$, then $(X, \mathcal{A}, \mu)$ is called a measure space.

Proposition 1.1 *Consider the measure space $(X, \mathcal{A}, \mu)$.*

1) *Let A and B be subsets of X, such that $A \subset B$. Then $\mu(A) \leq \mu(B)$.*

2) *If $\{A_k\}$ is a sequence of arbitrary subsets of X that belong to $\mathcal{A}$, then,* $\mu(\bigcup\limits_{k=1}^{\infty} A_k) \leq \sum\limits_{k=1}^{\infty} \mu(A_k)$.

3) ***a)*** *If $\{A_k\}$ is an increasing sequence of sets then $\mu(\bigcup\limits_{k=1}^{\infty} A_k) = \lim\limits_{k\to\infty} \mu(A_k)$.*

b) *If $\{A_k\}$ is a decreasing sequence of sets then $\mu(\bigcap\limits_{k=1}^{\infty} A_k) = \lim\limits_{k\to\infty} \mu(A_k)$.*

Proof (Roussas 2004).

Definition 1.3 *Let* $\mathrm{P}(X)$ *be the power set of* X. *An outer measure* $\mu^* : \mathrm{P}(X) \to [0, \infty]$ *is a function defined from* $\mathrm{P}(X)$ *to nonnegative extended real line that satisfies:*

a) $\mu^*(\emptyset) = 0$.

b) μ^* *is monotone i.e., if* $A \subseteq B \subset X$, *then* $\mu^*(A) \leq \mu^*(B)$.

c) $\mu^*(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \mu^*(A_k)$ *where* $\{A_k\}$ *is any infinite sequence of subsets of* X.

Definition 1.4 ***(Caratheodory)*** *Let* X *be a set and* μ^* *be an outer measure on* X. *A subset* B *of* X *is said to be* μ^**-measurable if for each subset* A *of* X,

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c) \tag{1.1}$$

holds where B^c *denotes the complement of* B, $B^c = X \setminus B$.

Proposition 1.2 *Let* $\boldsymbol{M}(\mu^*)$ *denote the family of all* μ^**-measurable sets of* X. *Then* $(X, \boldsymbol{M}(\mu^*), \mu^*)$ *is a measure space.*

Proof (Roussas 2004).

Definition 1.5 *Lebesgue outer measure on* $\mathbb{R}$ *is the measure defined on the semiring* $S = \{[a, b) : a, b \in \mathbb{R}, and a \leq b\}$ *by* $\lambda([a, b)) = b - a$ *and call* λ^* *the outer measure generated by* λ. *Lebesgue measure is the measure that assigns any interval its length.*
A subset I *of* $\mathbb{R}$ *is called an interval if for every* $x, y \in \mathbb{R}$ *with* $x < y$, *we have* $[x, y] \subset I$. *If* I *has any one of the forms* $[a, b)$, $[a, b]$, $(a, b]$ *or* (a, b) *with* $-\infty < a < b < \infty$, *then* I *is called a bounded interval and its length is defined by* $|I| = b - a$. *If* I *is not bounded, then its length is said to be infinite.*

Proposition 1.3 *Lebesque outer measure* λ^* *on* $\mathbb{R}$ *is translation invariant in the sense that if* $x \in \mathbb{R}$ *and* $A \subset \mathbb{R}$, *then* $\lambda^*(A) = \lambda^*(A + x)$.

Proof (Dönmez 2001).

Definition 1.6 *A Borel set is any set obtained from open sets by countably many operations, each operation consisting of taking unions, intersections or complement.*

Theorem 1.1 *Every Borel set is measurable. In particular, each open set and each closed set is measurable.*

Proof (Royden 1988).

Definition 1.7 *If μ^* is an outer measure of X, then a relation involving the elements of X is said to hold almost everywhere (or that it holds for almost all x), shortly a.e., if the set of all points for which the relation fails to hold is a null set.*

Definition 1.8 *An extended real-valued function f is said to be measurable if the set $\{x : f(x) > \alpha\}$ is measurable for each real number α.*

It should be noted that if f is a measurable function and E is a measurable subset of domain of f, then the function obtained by restricting f to E is also measurable.

Proposition 1.4 *Let $(X, \mathcal{A})$ be a measurable space, f and g be real valued measurable functions on $\mathcal{A}$, α be a real number. Then αf, $f + g$, $f - g$, fg, $\sup f_n$, $\inf f_n$, $\overline{\lim} f_n$, $\underline{\lim} f_n$ and $\dfrac{f}{g}$ are also measurable.*

Proof (Cohn 1997).

Proposition 1.5 *Let $(X, \mathcal{A})$ be a measurable space, let A be a subset of X that belongs to $\mathcal{A}$ and f be a $[0, +\infty]$-valued measurable function on X. Then, there exists a sequence $\{S_n\}$ of simple functions on X which satisfies*

$$S_1 \leq S_2 \leq \ldots \text{ and } f = \lim_{n\to\infty} S_n.$$

Proof (Rudin 1987).

Definition 1.9 *The function χ_A defined by*

$$\chi_A = \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad x \in E \setminus A \end{cases} \tag{1.2}$$

is called the characteristic function of E and a linear combination of the characteristic functions as

$$S = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \tag{1.3}$$

is called a simple function where A_is are pairwise disjoint sets with $A_i = \{t : S(t) = \alpha_i\}$, $i = 1, 2, \ldots, n$.

Definition 1.10 *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let E be a measurable set and S be a simple function that takes positive values α_i on the measurable sets A_i $i = 1, 2, ...n$ and is zero elsewhere. The integral of S is defined with respect to corresponding measure μ as*

$$\int_E S d\mu = \sum_{i=1}^{n} \alpha_i \mu(E \cap A_i). \tag{1.4}$$

Definition 1.11 *Let $(X, \mathcal{A}, \mu)$ be a measure space and f be a nonnegative measurable function defined on a measurable set E. The integral of f is defines as*

$$\int_E f d\mu = \sup \int_E S d\mu. \tag{1.5}$$

where the supremum is taken over all nonnegative simple functions S, such that $S \leq f$.

Definition 1.12 *Let $(X, \mathcal{A}, \mu)$ be a measure space and f be a measurable function which takes both positive and negative values defined on a measurable set E. The integral of f on E is defines as*

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu \tag{1.6}$$

where $f^+ = \max\{0, f\}$ and $f^- = \max\{0, -f\}$, provided that at least one of the integrals $\int_E f^+ d\mu$ or $\int_E f^- d\mu$ has finite value, otherwise $\int_E f d\mu$ is undefined. If both $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are finite, f is said to be integrable on E. When one of $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ is finite but not both, $\int_E f d\mu$ is said to exist but with infinite value.

Proposition 1.6 *Let $(X, \mathcal{A}, \mu)$ be a measure space, f, g be real-valued integrable functions on X, α be a real number. Then,*

a) *αf and $f + g$ are integrable.*

b) $\int \alpha f d\mu = \alpha \int f d\mu$.

c) $\int (f + g) d\mu = \alpha \int f d\mu + \alpha \int g d\mu$.

d) *If $f(x) \leq g(x)$ for $\forall x \in X$, then $\int f d\mu \leq \int g d\mu$.*

Proof (Cohn 1997).

Proposition 1.7 *Let $(X, \mathcal{A}, \mu)$ be a measure space, f and g be $[-\infty, +\infty]$ valued measurable functions on X with $f = g$ a.e.. If either $\int f d\mu$ or $\int g d\mu$ exists, then both exist with $\int f d\mu = \int g d\mu$.*

Proof (Cohn 1997).

1.2. Basic Concepts in Time Scales

The theory of time scale was introduced by Stefan Hilger in his Ph. D. thesis in 1988 which has supervised by Bernd Auldbach in order to unify continuous and discrete analysis. However, since there are many other time scales than just the set of real numbers or the set of integers, one has a much more general result so unification and extension can be given as two main features of the theory of time scale.

In this section, we will introduce the theory of time scale by giving some basic concepts which we will use in this study. We refer to (Bohner and Peterson 2001) for more information.

Definition 1.13 *A time scale is an arbitrary nonempty closed subset of the real numbers. Thus, $\mathbb{R}$ itself, $\mathbb{Z}$, $\mathbb{N}$, $h\mathbb{Z}$ where h is a positive constant, union of any closed intervals as $[0,1] \cup [2,3] \cup [8,9]$, or any closed intervals plus some single points as $[0,1] \cup \{4,6,8\}$ can be given as examples of time scales whereas $\mathbb{Q}$, $\mathbb{C}$, any subset of real numbers that is not closed or not union of closed intervals or single points are not time scales.*

Throughout this thesis we denote a time scale by the symbol $\mathbb{T}$. We will begin with three basic operators on $\mathbb{T}$, that is forward jump operator, backward jump operator and graininess function.

Definition 1.14 *Let $\mathbb{T}$ be a time scale. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by*

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \tag{1.7}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}. \tag{1.8}$$

If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$, we say t is left-scattered. If t is both right-scattered and left-scattered, we say t is isolated. Also, if $\sigma(t) = t$, then t is called right-dense and if $\rho(t) = t$, then t is called left-dense. If t is both right-dense and left-dense, then we say t is a dense point. For special cases if $t = \max\mathbb{T}$, $\sigma(t) = t$ and if $t = \min\mathbb{T}$, $\rho(t) = t$.

Definition 1.15 *The function $\mu : \mathbb{T} \to [0, \infty)$ is defined by*

$$\mu(t) := \sigma(t) - t. \tag{1.9}$$

is called graininess function.

Definition 1.16 *A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at all right-dense points and the left-sided limits exist on left-dense points.*

Definition 1.17 *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Here $\mathbb{T}^k$ is a new term derived from $\mathbb{T}$, if $\mathbb{T}$ has a left scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$. Then we define delta derivative $f^\Delta(t)$ to be the number provided it exists with the property that given any $\epsilon > 0$, there is a neighborhood U of t (i.e, $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s| \quad for \;\; all \;\; s \in U. \tag{1.10}$$

$f^\Delta(t)$ is called the delta derivative of f at t.

Moreover, we say that f is Δ-differentiable on $\mathbb{T}^k$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^\Delta : \mathbb{T}^k \to \mathbb{R}$ is then called the delta (Δ) derivative of f on $\mathbb{T}^k$.

Theorem 1.2 *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following:*

(i) *If f is differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) *If t is right-dense, then f is differentiable at t if and only if the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number, in this case

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) *If f is differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t)$.*

Proof (Bohner and Peterson 2001).

Definition 1.18 *A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated, provided its right-sided limit exists at all right-dense points in $\mathbb{T}$ and its left-sided limit exists at all left-dense points.*

Definition 1.19 *A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable with region of differentiation D, provided $D \subset \mathbb{T}^k$, $\mathbb{T}^k \backslash D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$.*

Theorem 1.3 *Let $f : \mathbb{T} \to \mathbb{R}$ be a regulated function. Then there exists a function F, which is pre-differentiable with region of differentiation D such that*

$$F^{\Delta}(t) = f(t) \quad \textit{holds for all} \quad t \in D.$$

Here F is called pre-antiderivative of f.

Proof (Bohner and Peterson 2001).

Definition 1.20 *Let $f : \mathbb{T} \to \mathbb{R}$ be a regulated function. We define an indefinite integral of a regulated function f by*

$$\int f(t)\Delta t = F(t) + c,$$

where c is an arbitrary constant and F is a pre-antiderivative of f. We define the Cauchy integral by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \textit{for all} \quad r, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$, provided

$$F^{\Delta}(t) = f(t) \quad \textit{holds for all} \quad t \in \mathbb{T}^k.$$

Theorem 1.4 *Let $f : \mathbb{T} \to \mathbb{R}$ be an rd-continuous function, $t_0 \in \mathbb{T}$ and $t \in \mathbb{T}^k$*

$$F(t) := \int_{t_0}^t f(s)\Delta s$$

is antiderivative of f.

Proposition 1.8 *If $f : \mathbb{T} \to \mathbb{R}$ is an rd continuous function and $t \in \mathbb{T}^k$, then*

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t).$$

Proof (Bohner and Peterson 2001).

CHAPTER 2

MEASURE ON TIME SCALE

It is necessary to generalize the geometric concept of "$length$" defined for intervals and the generalization is called measure, specifically delta (Δ) measure and nabla (∇) measure on time scales.

Measure theory on time scales was first constructed by Guseinov (Guseinov, 2003), (Guseinov and Bohner, 2003) then further studies were made by Cabada-Vivero (Cabada and Vivero, 2004)and Rzezuchowski (Rzezuchowski, 2005). In this chapter, we will first give how Δ-measure was constructed in detail and using the philosophy which is included by the related papers, we will adapt general concepts of measure theory to time scales.

2.1. Constructing Δ-Measure

Let $\mathbb{T}$ be a time scale, σ and ρ be the forward and the backward jump operators defined on $\mathbb{T}$ respectively as given in Definition 1.14. Let $\Im_1$ denote the class of all bounded left closed and right open intervals of $\mathbb{T}$ of the form

$$[a,b) = \{t \in \mathbb{T} : a \leq t < b, \quad a,b \in \mathbb{T}\}. \tag{2.1}$$

When $a = b$, the interval reduces to the empty set. On the class of $\Im_1$ of semi-closed intervals, a set function $m_1 : \Im_1 \to [0,+\infty]$ is defined which assigns each interval $[a,b)$ its length i.e.,

$$[a,b) \to m_1\big([a,b)\big) = b - a. \tag{2.2}$$

The set function m_1 is countably additive measure on $\Im_1$. Indeed,

1) $m_1([a,b)) = b - a \geq 0$ since $b \geq a$.

2) $m_1\big(\bigcup_{i=1}^{\infty}[a_i,b_i)\big) = \sum_{i=1}^{\infty} m_1\big([a_i,b_i)\big)$ for pairwise disjoint intervals $[a_i,b_i)$.

3) $m_1(\emptyset) = m_1([a,a)) = a - a = 0$ holds for any $a \in \mathbb{T}$.

Caratheodory extension μ_Δ of m_1 is described by following this procedure: First outer measure is defined on all subsets of $\mathbb{T}$ by using m_1, then some of these sets which satisfies countably additivity is picked.

Definition 2.1 *Let* E *be any subset of* $\mathbb{T}$. *If there exists at least one finite or countable system of intervals* $I_j \in \Im_1$ $(j = 1, 2, ...)$ *such that* $E \subset \bigcup_j I_j$, *then* $m_1^*(E) = \inf \sum_j m_1(I_j)$ *is called the outer measure of* E, *where the infimum is taken over all coverings of* E *by a finite or countable system of intervals* $I_j \in \Im_1$.

Definition 2.2 *A property that holds everywhere except for a null set is said to hold* Δ-*almost everywhere, shortly* Δ-*a.e. in measure theory on time scale.*

The outer measure is always nonnegative but could be infinite so that in general we have $0 \le m_1^*(E) \le \infty$. In case there is no such covering of E, we say E is not coverable by finite or countable system of intervals and the outer measure of this set is equal to infinity, i.e., $m_1^*(E) = \infty$.

It is the problem although m_1^* is defined for any subset of $\mathbb{T}$, it is not countable (even finite) additive. Our aim is to find a σ-algebra where m_1^* is countably additive and contains the algebra on which m_1 is defined. The condition which was given by Caratheodory exhibits the required restriction for this. In the following defnition, this restriction is adapted to time scales.

Definition 2.3 *A set* $E \subset \mathbb{T}$ *is called* m_1^*-*measurable or* Δ-*measurable if for each interval* $I \subset \Im_1$

$$m_1^*(I) = m_1^*(I \cap E) + m_1^*(I \cap E^c) \tag{2.3}$$

where $E^c = \mathbb{T} - E$.

The next proposition will give us a more flexible criterion for Δ-measurability of a set in a time scale.

Proposition 2.1 *If* E *is a* Δ-*measurable set, then for any set* $A \subset \mathbb{T}$

$$m_1^*(A) = m_1^*(A \cap E) + m_1^*(A \cap E^c) \tag{2.4}$$

holds for all $A \in \mathbb{T}$ *whether* A *is* Δ-*measurable or not.*

Proof As in (Craven 1982).

This identity is the desired property of additivity for the outer measure m_1^*. Roughly, this equality shows E and E^c are disjoint enough in order to divide any set A in the sense of additivity.(Bartle 1992)

Proposition 2.2 *The family of all m_1^*-measurable subsets denoted by $\boldsymbol{M}(m_1^*)$ of $\mathbb{T}$ forms a σ-algebra.*

Proof

a) For $\forall E \in \mathbf{M}(m_1^*)$ we need to prove first $E^c \in \mathbf{M}(m_1^*)$. Setting $E = (E^c)^c$, Equation 2.4 holds, that is,

$$m_1^*(A \cap (E^c)^c) + m_1^*(A \cap E^c) = m_1^*(A \cap E) + m_1^*(A \cap E^c) = m_1^*(A)$$

for any $A \subset \mathbb{T}$, which implies $E^c \in \mathbf{M}(m_1^*)$.

b) Let E_1 and E_2 be disjoint m_1^*-measurable sets on $\mathbb{T}$. We need to prove if $E_1 \cup E_2$ is m_1^*-measurable or not. For all $A \subset \mathbb{T}$

$$m_1^*(E_1 \cup E_2) < m_1^*(A \cap (E_1 \cup E_2)) + m_1^*(A \cap (E_1 \cup E_2)^c)$$

is obvious. Using the subadditivity property of outer measure and rewriting the right hand side of the inequality

$$\begin{aligned} m_1^*(E_1 \cup E_2) &< m_1^*((A \cap E_1) - E_2)) + m_1^*((A \cap E_1) \cap E_2)) \\ &\quad + m_1^*((A - E_1) \cap E_2)) + m_1^*((A - E_1) - E_2)) \\ &= m_1^*(A \cap E_1) + m_1^*(A - E_1) \\ &= m_1^*(A) \end{aligned}$$

which implies

$$m_1^*(E_1 \cup E_2) = m_1^*(A \cap (E_1 \cup E_2)) + m_1^*(A \cap (E_1 \cup E_2)^c).$$

We have shown that $\mathbf{M}(m_1^*)$ is an algebra. Now, let $E = \bigcup_{j=1}^{\infty} E_j$ where $E_j \in \mathbf{M}(m_1^*)$.

$$\bigcup_{j=1}^{\infty} E_j = E_1 \cup (E_1^c \cap E_2) \cup (E_1^c \cap E_2^c \cap E_3) \cup ...$$

and let

$$B_n = \bigcup_{j=1}^{n} E_j, \quad B_0 = \emptyset.$$

Then $B_n \in \mathbf{M}(m_1^*)$, that is, for any $A \subset \mathbb{T}$, $m_1^*(A) = m_1^*(A \cap B_n) + m_1^*(A \cap B_n^c)$,

$$\begin{aligned} m_1^*(A \cap B_n) &= m_1^*((A \cap B_n) \cap E_n) + m_1^*((A \cap B_n) \cap E_n^c) \\ &= m_1^*(A \cap E_n) + m_1^*(A \cap B_{n-1}). \end{aligned}$$

Working in the same way

$$m_1^*(A \cap B_n) = \sum_{k=1}^{n} m_1^*(A \cap E_k).$$

Thus

$$\begin{aligned} m_1^*(A) &= \sum_{k=1}^{n} m_1^*(A \cap E_k) + m_1^*(A \cap B_n^c) \\ &\geq \sum_{k=1}^{n} m_1^*(A \cap E_k) + m_1^*(A \cap E^c) \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} m_1^*(A) &\geq \sum_{k=1}^{\infty} m_1^*(A \cap E_k) + m_1^*(A \cap E^c) \\ &\geq m_1^*(\bigcup_{k=1}^{\infty}(A \cap E_k)) + m_1^*(A \cap E^c) \\ &\geq m_1^*(A \cap (\bigcup_{k=1}^{\infty} E_k)) + m_1^*(A \cap E^c) \\ &= m_1^*(A \cap E) + m_1^*(A \cap E^c) \geq m_1^*(A) \end{aligned}$$

Whence $\mathbf{M}(m_1^*)$ forms a σ-algebra. $\square$

Caratheodory extension theorem says that shortly for given m_1 can be extended to m_1^* defined on the unique σ-algebra that contains the σ-algebra on which m_1 is defined.

The family of all m_1^*-measurable sets is a σ-field and m_1^* restricted to that σ-field is a countably additive measure denoted by μ_Δ, that is, measure constructed on time scales.

Proposition 2.3 ***(Increasing Sequence Theorem and Decreasing Sequence Theorem)*** *If $\{E_n\}$ is an increasing sequence of sets in $\mathbb{T}$, then*

$$\mu_\Delta(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu_\Delta(E_n). \tag{2.5}$$

Similarly, if $\{E_n\}$ is a decreasing sequence of sets in $\mathbb{T}$, then

$$\mu_\Delta(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu_\Delta(E_n). \tag{2.6}$$

In other words, Δ-measure on $\mathbb{T}$ is continuous.

Proof Let $A_n \in \mathbf{M}(m_1^*)$ and $\{A_n\}_{n\in\mathbb{N}}$ be a nondecreasing sequence of sets. If $\mu_\Delta(A_j) = \infty$ for some j, $\mu_\Delta(A_n) = \infty$ for all $n \geq j$, $\mu_\Delta(A_n) \to \mu_\Delta(\bigcup_{n=1}^{\infty} A_n) = \infty$ so we may assume that $\mu_\Delta(A_n) < \infty$ for all n. Then,

$$\begin{aligned}\bigcup_{n=1}^{\infty} A_n &= A_1 \cup (A_1^c \cap A_2) \cup ... \cup (A_{n-1}^c \cap A_n) \cup ... \\ &= A_1 \cup (A_2 \setminus A_1) \cup ... \cup (A_n \setminus A_{n-1}) \cup ...\end{aligned}$$

Thus

$$\begin{aligned}\mu_\Delta(\bigcup_{n=1}^{\infty} A_n) &= \mu_\Delta(A_1 \cup (A_1^c \cap A_2) \cup ... \cup (A_{n-1}^c \cap A_n) \cup ...) \\ &= \mu_\Delta(A_1 \cup (A_2 \setminus A_1) \cup ... \cup (A_n \setminus A_{n-1}) \cup ...) \\ &= \mu_\Delta(A_1) + \mu_\Delta(A_2 \setminus A_1) + ...\mu_\Delta(A_n \setminus A_{n-1}) + ... \\ &= \lim_{n\to\infty} (\mu_\Delta(A_1) + \mu_\Delta(A_2 \setminus A_1) + ... + \mu_\Delta(A_n \setminus A_{n-1})) \\ &= \lim_{n\to\infty} (\mu_\Delta(A_1) + \mu_\Delta(A_2) - \mu_\Delta(A_1) + ... + \mu_\Delta(A_n) - \mu_\Delta(A_{n-1})) \\ &= \lim_{n\to\infty} \mu_\Delta(A_n).\end{aligned}$$

This establishes continuity from below which plays an essential role in the theory of Lebesgue integration.

Now, let $\{A_n\}_{n\in\mathbb{N}}$ be a non-increasing sequence of sets as $n \to \infty$ and let $\mu_\Delta(A_{n_0}) < \infty$. Then $A_{n_0} - A_n$ is increasing for $n > n_0$ and $\bigcup_{n=n_0}^{\infty} (A_{n_0} - A_n) = A_{n_0} - \bigcap_{n=n_0}^{\infty} A_n$ i.e., as $n \to \infty$, $A_{n_0} - A_n \uparrow A_{n_0} - \bigcap_{n=n_0}^{\infty} A_n$. Thus

$$\mu_\Delta(A_{n_0} - A_n) \uparrow \mu_\Delta(A_{n_0} - \bigcap_{n=n_0}^{\infty} A_n) \text{by Equation 2.5.}$$

Since

$$\mu_\Delta(A_{n_0} - A_n) = \mu_\Delta(A_{n_0}) - \mu_\Delta(A_n)$$

and

$$\mu_\Delta(A_{n_0} - \bigcap_{n=n_0}^{\infty} A_n) = \mu_\Delta(A_{n_0}) - \mu_\Delta(\bigcap_{n=n_0}^{\infty} A_n),$$

we have

$$\mu_\Delta(A_n) \downarrow \mu_\Delta(\bigcap_{n=n_0}^{\infty} A_n) = \mu_\Delta(\bigcap_{n=1}^{\infty} A_n). \qquad \square$$

The hypothesis that $\mu_\Delta(A_1) < \infty$ can not be omitted from this proposition. A counter example can be given as follows:

Consider the Lebesgue measure on $\mathbb{R}$, then the corresponding measure is the classical Lebesgue measure. For $A_n = [n, \infty) \subset \mathbb{T} = \mathbb{R}$, $\{A_n\} \downarrow \emptyset$ whereas $\mu_\Delta(A_n) = \lambda(A_n) = \infty \not\to \mu_\Delta(\emptyset) = \lambda(\emptyset) = 0$. In order to avoid this contradiction, we assume that there exists $j \in \mathbb{N}$ such that $\mu_\Delta(A_j)$ is finite, then for all $n \geq j$, $\mu_\Delta(A_n)$s are also finite.

Lemma 2.1 *Any single point set $\{t_0\} \subset \mathbb{T}$ is m_1^*-measurable.(Rzezuchowski 2005)*

Proof We have to prove that for every $A \subset \mathbb{T}$, the following equality holds:

$$m_1^*(A) = m_1^*(A \cap \{t_0\}) + m_1^*(A \backslash \{t_0\}). \tag{2.7}$$

It is obvious when $t_0 \notin A$, so let $t_0 \in A$. It is easy to see that if $\max \mathbb{T} \in A$, Equality 2.7 holds. Let $A \subset \mathbb{T} - \{\max \mathbb{T}\}$. In this case

$$m_1^*(A) \leq m_1^*(A \cap \{t_0\}) + m_1^*(A \backslash \{t_0\}) \tag{2.8}$$

is always true so we only need to prove the opposite side. Let $\bigcup_{i \in I}[a_i, b_i)$, $I \subset \mathbb{N}$ be a covering of A as

$$m_1^*(A) = \inf\{\sum_{i \in I}(b_i - a_i) : A \subset \bigcup_{i \in I}[a_i, b_i); a_i, b_i \in \mathbb{T}, a_i < b_i, I \subset \mathbb{N}\}. \tag{2.9}$$

We define a covering of $A \backslash \{t_0\}$ and another of $\{t_0\}$. If $t_0 \notin [a_i, b_i)$, then we include this interval to the covering of $A \backslash \{t_0\}$. If $t_0 \in [a_i, b_i)$, then we include its subinterval $[t_0, \sigma(t_0))$ into the covering of $\{t_0\}$ and if any of intervals $[a_i, t_0)$ or $[\sigma(t_0), b_i)$ is non-empty, we include it to the covering of $A \backslash \{t_0\}$. Considering all $[a_i, b_i)$, we actually get the required two coverings. Moreover, the sum of lengths of intervals in both coverings is equal to $\sum_{i \in I}(b_i - a_i)$ which finally implies

$$m_1^*(A) \geq m_1^*(A \cap \{t_0\}) + m_1^*(A \backslash \{t_0\}). \qquad \square$$

Suppose that $\mathbb{T}$ has a finite maximum τ_0. Obviously the set $X = \mathbb{T} - \{\tau_0\}$ can be represented as a finite or countable union of intervals of the family $\Im_1$ and therefore it is Δ-measurable. Furthermore, the single point set $\{\tau_0\} = \mathbb{T} - X$ is Δ-measurable as the difference of two Δ-measurable sets $\mathbb{T}$ and X but $\{\tau_0\}$ does not have a finite or countable covering intervals of $\Im_1$, therefore, the single point set $\{\tau_0\}$ and also any Δ-measurable subset of $\mathbb{T}$ containing τ_0 have Δ-measure infinity.

Theorem 2.2 *Δ-measure of a single point set $\{t_0\} \subset \mathbb{T} - \{max\mathbb{T}\}$ is given by*

$$\mu_\Delta(\{t_0\}) = \sigma(t_0) - t_0 = \mu(t_0) \tag{2.10}$$

where μ denotes the Grainess function given in Definition 1.15. (Guseinov 2003)

Proof If t_0 is right-scattered, $\{t_0\} = [t_0, \sigma(t_0)) \in \Im_1$. Therefore $\{t_0\}$ is Δ-measurable and we have

$$\mu_\Delta(\{t_0\}) = m_1([t_0, \sigma(t_0))) = \sigma(t_0) - t_0 \tag{2.11}$$

which is the desired result. For other words, if we have any covering of $\{t_0\}$, then every interval containing t_0 must also include $[t_0, \sigma(t_0))$ and on the other hand this last interval itself is the smallest covering of $\{t_0\}$.

Now, consider the case when t_0 is right-dense. In this case there exists a decreasing sequence of sets such that $\{t_k\}$ of points of $\mathbb{T}$ such that $[t_0, t_1) \supset [t_0, t_2) \supset ...$ and $t_k > t_0$ and $t_k \to t_0$, $\{t_0\} = \bigcap_{k=1}^{\infty} [t_0, t_k)$. Hence $\{t_0\}$ is Δ-measurable as a countable intersection of Δ-measurable sets and by Proposition 2.3 we have

$$\mu_\Delta(\{t_0\}) = \lim_{k\to\infty} \mu_\Delta([t_0, t_k)) = \lim_{k\to\infty} (t_k - t_0) = 0 \tag{2.12}$$

and so the equation 2.10 holds for this case as well. $\square$

The following lemma permits us to use this fact to other specific sets in $\mathbb{T}$.

Lemma 2.3 *The set of all right-scattered points of $\mathbb{T}$ is at most countable, that is, there are $\{t_i\}_{i\in I} \subset \mathbb{T}$, $I \subset \mathbb{N}$ such that*

$$\mathbb{T} \setminus D_R = S_R = \{t_i\}_{i\in I} \tag{2.13}$$

where D_R denotes the all right-dense points and S_R denotes the all right-scattered points of $\mathbb{T}$.(Cabada and Vivero 2004)

Proof Let $g : \mathbb{T}^\sim \to \mathbb{R}$ be defined as

$$g(t) = \begin{cases} t & if \quad t \in \mathbb{T} \\ t_i & if \quad t \in (t_i, \sigma(t_i)). \end{cases}$$

Here, $\mathbb{T}^\sim$ is obtained by filling the blanks $(t_i, \sigma(t_i))$ of $\mathbb{T}$ where $t_i < \sigma(t_i)$. It is obvious that the function g is monotone increasing on $\mathbb{T}^\sim$. Since the set of points where a monotone function has discontinuities is at most countable (see (Carter and Brunt 2000)), we arrive the desired result. $\square$

Remark 2.4 *From now and on, we will express the extension of a set* $E \subset \mathbb{T}$ *as* $E^{\sim} = E \cup \bigcup_{i \in I_E} (t_i, \sigma(t_i))$ *where* I_E *denotes the indices set of right-scattered points of* E*.*

Corollary 2.5 *The sets* S_R *and* D_R *introduced in Lemma 2.3 are both* Δ*-measurable.*

Proof S_R is a countable union of single point sets, so Δ-measurable. Consequently, D_R is Δ-measurable as the difference of two Δ-measurable sets $\mathbb{T}$ and S_R. □

We have proved that a single point subset of $\mathbb{T}$ is Δ-measurable for the first step in order to show all types of intervals are also Δ-measurable. We will use the fact that by adding or subtracting endpoints to $[a, b)$, which we exactly know Δ-measurable, we obtain other types of intervals are Δ-measurable.

The next theorem, which was established by Guseinov, gives Δ-measure of any interval by using this fact.

Theorem 2.6 *If* $a, b \in \mathbb{T}$ *and* $a \leq b$*, then*

a) $\mu_\Delta\big([a, b)\big) = b - a$*.*

b) $\mu_\Delta\big((a, b)\big) = b - \sigma(a)$*.*

If $a, b \in \mathbb{T} - \{\max \mathbb{T}\}$ *and* $a \leq b$*, then*

c) $\mu_\Delta\big((a, b]\big) = \sigma(b) - \sigma(a)$*.*

d) $\mu_\Delta\big([a, b]\big) = \sigma(b) - a$*.(Guseinov 2003)*

Proof

a) All intervals of the family $\Im_1$ are Δ-measurable so the first formula is obvious since

$$\mu_\Delta([a, b)) = m_1([a, b)) = b - a.$$

b) In order to prove the second one we write $[a, b) = \{a\} \cup (a, b)$. Hence, by additivity,

$\mu_\Delta([a, b)) = \mu_\Delta(\{a\} \cup (a, b))$

$\mu_\Delta([a, b)) = \mu_\Delta(\{a\}) + \mu_\Delta((a, b))$

$$\begin{aligned} \mu_\Delta((a, b)) &= \mu_\Delta([a, b)) - \mu_\Delta(\{a\}) \\ &= b - a - (\sigma(a) - a) \\ &= b - \sigma(a). \end{aligned}$$

c) In order to prove this assertion, we can use **b)** with the identity $(a, b] = (a, b) \cup \{b\}$ as

$$\begin{aligned}\mu_\Delta((a,b]) &= \mu_\Delta((a,b) \cup \{b\}) \\ &= \mu_\Delta((a,b)) + \mu_\Delta(\{b\}) \\ &= b - \sigma(a) + \sigma(b) - b \\ &= \sigma(b) - \sigma(a).\end{aligned}$$

d) For this case we can use **c)** with the identity $[a, b] = \{a\} \cup (a, b]$.

$$\begin{aligned}\mu_\Delta([a,b]) &= \mu_\Delta(\{a\} \cup (a,b]) \\ &= \mu_\Delta(\{a\}) + \mu_\Delta((a,b]) \\ &= \sigma(a) - a + \sigma(b) - \sigma(a) \\ &= \sigma(b) - a. \qquad \square\end{aligned}$$

Remark 2.7 *An important property of Lebesgue measure in $\mathbb{R}$ is its translation invariance, given in Definition 1.3. Nevertheless, it is not satisfied for Δ-measure in general except for some typical time scales. First of all, we need to restrict our attention to the subsets of $\mathbb{T} - \{\max \mathbb{T}\}$. If the graininess function μ is a function which has different values for distinct points, Δ-measure of a subset $E \subset \mathbb{T} - \{\max \mathbb{T}\}$ does not coincide with the Δ-measure of the subset $E_k \subset \mathbb{T} - \{\max \mathbb{T}\}$ that is obtained by shifting E k units for all possible k values since Δ-measure of a set is dependent to μ and differs from point to point.*

However, there are some special cases. For instance, taking $\mu(t) = h$ with h is any positive constant, corresponding time scale is unique i.e., $\mathbb{T} = h\mathbb{Z}$, Δ-measure is translation invariant. For this case, time scale is countable since there is one to one correspondence with integers. Here, we can use the fact that for all subsets of $\mathbb{T}$ can be written as the union of countable number of single point sets with Δ-measure h, thus for a set with n element has Δ-measure nh. Shifting the set $k = hr$ units where r is any integer, there is no change for the value of measure because a shifted set can be represented as union of the set of shifted points and Δ-measure of each shifted single point set stays the same, so does the Δ-measure of the shifted set which containing them. Briefly, Δ-measure is translation invariant for the case $\mathbb{T} = h\mathbb{Z}$ where h is any positive real number.

Remark 2.8 *It should be taken into account that although for an arbitrary subset $A \subset \mathbb{R}$ and the set $aA + b = \{at + b : t \in A\}$,*

i) $\lambda^*(aA+b) = |a|\lambda^*(A)$,

ii) *If A is Lebesgue measurable, so is $aA+b$*

do not hold for an arbitrary time scale. First of all, since $A \subset \bigcup_{i=1}^{\infty}[a_i, b_i)$ corresponds $A+b \subset \bigcup_{i=1}^{\infty}[a_i+b, b_i+b)$ for this case, it is easy to see that $\lambda^*(A+b) = \lambda^*(A)$ which implies translation invariance. Thus, as it has been given in Remark 2.7, translation invariance is only satisfied for special cases, i.e., $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$ where h is a positive constant.

For instance, let us consider a special time scale $\mathbb{T} = h\mathbb{Z}$ where h is any positive constant.

$$m_1^*(A) = m_1^*(\bigcup_{i=1}^{n}\{a_i\}) = \sum_{i=1}^{n} m_1^*(\{a_i\}) = \sum_{i=1}^{n} h = nh.$$

For a constant $k = hr$,

$$m_1^*(kA) = m_1^*(\bigcup_{i=1}^{n}\{ka_i\}) = \sum_{i=1}^{n} m_1^*(\{ka_i\}) = \sum_{i=1}^{n} h = nh.$$

2.2. Relation Between Lebesgue Measure and Lebesgue Δ-Measure

Before generalizing the relations, it is convenient to exhibit that a time scale $\mathbb{T}$ is Lebesgue measurable. In fact it is not difficult to see that any time scale is a Borel set (see Definition 1.6) and so Lebesgue measurable from Theorem 1.1.

Now, let us compare two outer measures, Lebesgue outer measure λ^* defined on $\mathbb{R}$ and the outer measure m_1^* defined on $\mathbb{T}$. By λ, we mean the usual Lebesgue measure on $\mathbb{R}$ and λ^* the corresponding outer measure, that is,

$$\lambda^*(A) = \inf\{\sum_{j\in J}(\beta_j - \alpha_j) : A \subset \bigcup_{j\in J}[\alpha_j, \beta_j] \ \ , \alpha_j, \beta_j \in \mathbb{R}, \ \ \alpha_j \le \beta_j, J \subset \mathbb{N}\}. \quad (2.14)$$

Lemma 2.9 *If $E \subset \mathbb{T} - \{\max \mathbb{T}\}$, then the following properties are satisfied:*

i) $m_1^*(E) \ge \lambda^*(E)$.

ii) *If E does not include right-scattered points, then $m_1^*(E) = \lambda^*(E)$.*

iii) *The sets D_R and $S_R = \mathbb{T} \backslash D_R$ are Lebesgue measurable and $\lambda(S_R) = 0$ but*

$$\mu_\Delta(E \cap S_R) = \sum_{i \in I_E}(\sigma(t_i) - t_i)$$

where I_E indicates the indices set for all right-scattered points in E.

iv) $m_1^*(E) = \sum_{i \in I_E} (\sigma(t_i) - t_i) + \lambda^*(E)$.

v) $m_1^*(E) = \lambda^*(E)$ *if and only if E has no right-scattered points.*

(Cabada and Vivero 2004, Rzeuchowski 2005)

Proof

i) If one compares these two following equalities

$$\lambda^*(E) = \inf\{\sum_{j \in J} (\beta_j - \alpha_j) : E \subset \bigcup_{j \in J} [\alpha_j, \beta_j],\ \alpha_j \leq \beta_j,\ J \subset \mathbb{N}\} \tag{2.15}$$

$$m_1^*(E) = \inf\{\sum_{i \in I} (b_i - a_i) : E \subset \bigcup_{i \in I} [a_i, b_i),\ a_i \leq b_i,\ I \subset \mathbb{N}\} \tag{2.16}$$

then can see that every interval $[a_i, b_i)$ which may arise in the definition of $m_1^*(E)$ is also among intervals $[a_i, b_i]$ which appear in the definition of $\lambda^*(E)$. So the infimum defined by $\lambda^*(E)$ is surely less than or equal to the one defined by $m_1^*(E)$ thus $\lambda^*(E) \leq m_1^*(E)$ is showed.

ii) Since $\lambda^*(E) \leq m_1^*(E)$ has been proved in **i)**, the only relation we need to prove is the opposite one i.e., $\lambda^*(E) \geq m_1^*(E)$.

Suppose $\{[\alpha_j, \beta_j] : j \in J,\ J \subset \mathbb{N}\}$ is any covering of E. We fix $\varepsilon > 0$ and prove that there exists a covering $\{[a_i, b_i), i \in I,\ I \subset \mathbb{N}\}$ such that

$$m_1^*(E) = \inf\{\sum_{i \in I} (b_i - a_i) : E \subset \bigcup_{i \in I} [a_i, b_i); a_i, b_i \in \mathbb{T}, a_i < b_i, I \subset \mathbb{N}\}$$

with

$$\sum_{i \in I} (b_i - a_i) \leq \sum_{j \in J} (\beta_j - \alpha_j) + \varepsilon. \tag{2.17}$$

We eliminate first all intervals $[\alpha_j, \beta_j]$ which do not intersect $\mathbb{T} - \{\max \mathbb{T}\}$ and denote the remaining set of indices by I. For $i \in I$, we put $a_i = \sigma(\alpha_i)$ and for b_i, we consider two cases:

1) If $\beta_i \notin \mathbb{T}$ we put $b_i = \rho(\beta_i)$ then $b_i \notin E$ in view of the inclusion $E \subset D_R$.

2) If $\beta_i \in \mathbb{T}$, we have two possibilities:

a) $\beta_i \notin E$, we put $b_i = \beta_i$.

b) $\beta_i \in E$, we pick b_i from $(\beta_i, \beta + \frac{\varepsilon}{2^i}) \cap \mathbb{T}$, it is possible as β_i is not scattered. Due to the construction we have $a_i, b_i \in \mathbb{T}$ for all $i \in I$, $E \subset \bigcup_{i\in I}[a_i, b_i)$ and we obtain Inequality 2.17. The choice of $\varepsilon > 0$ was arbitrary, so $m_1^*(A) \leq \lambda^*(A)$, the proof is complete.

iii) Let $t_k \in \mathbb{T} - \{\max \mathbb{T}\}$. Then $\{t_k\} \subseteq [t_k - \epsilon, t_k + \epsilon)$ holds for each $\epsilon > 0$ and so

$$\lambda^*\{t_k\} \subseteq \lambda^*\big([t_k - \epsilon, t_k + \epsilon)\big) = 2\epsilon$$

and since ϵ is arbitrary, $\lambda^*(\{t_k\}) = 0$ which implies $\{t_k\}$ is Lebesgue measurable with measure zero. Since the set of all right-scattered points of any time scale is at most countable,

$$\lambda(\bigcup_{i\in I_{\mathbb{T}}}\{t_k\}) = \sum_{i\in I_{\mathbb{T}}}\lambda(\{t_k\}) = 0.$$

On the other hand,

$$\mu_\Delta(E \cap S_R) = \sum_{i\in I_E}(\sigma(t_i) - t_i) \leq (b - a) = \mu_\Delta([a, b))$$

is obvious from Lemma 2.1 we are able to replace m_1^* with μ_Δ.

vi) Suppose that $E \subset \mathbb{T} - \{\max \mathbb{T}\}$. Since S_R is Lebesgue measurable with $\lambda(S_R) = 0$,

$$\lambda^*(E) = \lambda^*(E \cap (S_R \cup (\mathbb{R}\backslash S_R))) = \lambda^*(E \cap S_R) + \lambda^*(E \cap (\mathbb{R}\backslash S_R)).$$

Since $E \subset \mathbb{T}$,

$$\lambda^*(E) = \lambda^*(E \cap S_R) + \lambda^*(E \cap (\mathbb{T}\backslash S_R))$$

Furthermore $E \cap S_R \subset S_R$ and $\lambda(S_R) = 0$, we obtain $\lambda(E \cap S_R) = 0$, finally we get

$$\lambda^*(E) = \lambda^*(E \cap (\mathbb{T}\backslash S_R)).$$

On the other hand, by Corollary 2.5, we write

$m_1^*(E) = m_1^*(E \cap S_R) + m_1^*(E \cap (\mathbb{T}\backslash S_R))$. Here $E \cap (\mathbb{T}\backslash S_R) \subset D_R$, hence $m_1^*(E \cap (\mathbb{T}\backslash S_R)) = \lambda^*(E \cap (\mathbb{T}\backslash S_R)) = \lambda^*(E)$ from **ii)**.

Thus

$$m_1^*(E) = \sum_{i\in I_E}(\sigma(t_i) - t_i) + \lambda^*(E).$$

v) Combining **ii)** and **iv)**, the proof is completed. □

Proposition 2.4 *Let $E \subset \mathbb{T}$. Then E is Lebesgue Δ-measurable if and only if it is Lebesgue measurable. In such a case, for $E \subset \mathbb{T} - \{\max \mathbb{T}\}$ the following is true:*

i) $\mu_\Delta(E) = \sum_{i\in I_E}(\sigma(t_i) - t_i) + \lambda(E).$

ii) $\lambda(E) = \mu_\Delta(E)$ *if and only if E has no right-scattered point (Cabada and Vivero 2004).*

Proof Suppose that $E \subset \mathbb{T}$ is Δ-measurable.

a) Let $A, E \subset \mathbb{T} - \{\max \mathbb{T}\}$. Using the identity $\mathbb{R} \setminus E = (\mathbb{T} \setminus E) \cup (\mathbb{R} \setminus \mathbb{T})$,

$$\lambda^*(A) \leq \lambda^*(A \cap E) + \lambda^*(A \cap (\mathbb{R} \backslash E)) \text{ turns into} \tag{2.18}$$

$$\begin{aligned} \lambda^*(A) &\leq \lambda^*(A \cap E) + \lambda^*(A \cap [(\mathbb{T} \setminus E) \cup (\mathbb{R} \setminus \mathbb{T})]) \\ &\leq \lambda^*(A \cap E) + \lambda^*(A \cap (\mathbb{T} \setminus E)) + \lambda^*(A \cap (\mathbb{R} \setminus \mathbb{T})) \\ &= \lambda^*(A \cap E) + \lambda^*(A \cap (\mathbb{T} \setminus E)) \text{ since } A \cap (\mathbb{R} \setminus \mathbb{T}) = \emptyset. \end{aligned}$$

Here, $\lambda^*(A \cap E) = m_1^*(A \cap E) - \sum_{i \in I_{A \cap E}} (\sigma(t_i) - t_i)$,

$$\lambda^*(A \cap (\mathbb{T} \backslash E)) = m_1^*(A \cap (\mathbb{T} \backslash E)) - \sum_{i \in I_{A \cap (\mathbb{T} \backslash E)}} (\sigma(t_i) - t_i).$$

Whence

$$\begin{aligned} \lambda^*(A) &\leq m_1^*(A \cap E) - \sum_{i \in I_{A \cap E}} (\sigma(t_i) - t_i) \\ &+ m_1^*(A \cap (\mathbb{T} \backslash E)) - \sum_{i \in I_{A \cap (\mathbb{T} \backslash E)}} (\sigma(t_i) - t_i). \end{aligned}$$

Since E is Δ-measurable, we have

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T} \backslash A))$$

and the relation

$$\sum_{i \in I_{E \cap A}} (\sigma(t_i) - t_i) + \sum_{i \in I_{E \cap (\mathbb{T} \backslash A)}} (\sigma(t_i) - t_i) = \sum_{i \in I_{E \cap \mathbb{T}}} (\sigma(t_i) - t_i).$$

Inequality 2.18 turns into

$$\lambda^*(E) \leq m_1^*(E) - \sum_{i \in I_{E \cap \mathbb{T}}} (\sigma(t_i) - t_i) = \lambda^*(E) \tag{2.19}$$

Finally, from 2.18 and 2.19, we obtain

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap (\mathbb{R} \backslash E))$$

so E is also Lebesgue measurable.

b) Let $E \subset \mathbb{T} - \{\max \mathbb{T}\}$, A contains the maximum point τ_0 of $\mathbb{T}$. We have

$$\lambda^*(A) \leq \lambda^*(A \cap E) + \lambda^*(A \cap (\mathbb{R} \setminus E)) \tag{2.20}$$

Let $A = B \cup \{\tau_0\}$. Then Inequality 2.20 turns into

$$\begin{aligned}
\lambda^*(A) &\leq \lambda^*((B \cup \{\tau_0\}) \cap E) + \lambda^*((B \cup \{\tau_0\}) \cap (\mathbb{R} \setminus E)) \\
&\leq \lambda^*(B \cap E) + \lambda^*(\{\tau_0\} \cap E) + \lambda^*(B \cap (\mathbb{R} \setminus E)) + \lambda^*(\{\tau_0\} \cap (\mathbb{R} \setminus E)) \\
&\leq \lambda^*(B \cap E) + \lambda^*(B \cap (\mathbb{R} \setminus E)) + \lambda^*(\{\tau_0\}) \\
&= \lambda^*(B \cap E) + \lambda^*(B \cap (\mathbb{R} \setminus E)).
\end{aligned}$$

Then the same procedure is followed as **i)** to show E is Lebesgue measurable.

c) Now suppose that E contains the maximum point τ_0 of $\mathbb{T}$. It is obvious that if E is Δ-measurable, then $E \setminus \{\tau\}$ is Δ-measurable, so $E \setminus \{\tau\}$ is Lebesgue measurable too. Since a single point set $\{\tau_0\}$ is Lebesgue measurable and using the fact that the union of two Lebesgue measurable sets is Lebesgue measurable, $(E \setminus \{\tau_0\}) \cup \{\tau_0\} = E$ is Lebesgue measurable as well.

Let E be Lebesgue measurable, then it is Δ-measurable and the proof follows similarly. $\square$

Remark 2.10 *By using the identities given by Proposition 2.4, we get*

$\mu_\Delta(E) = \lambda(E^\sim)$ *where* $E \subset \mathbb{T} - \{\max \mathbb{T}\}$

and $E^\sim$ *the extension of* E *introduced in Remark 2.4.*

Proof Consider the identity:

$$\begin{aligned}
\mu_\Delta(E) &= \sum_{i \in I_E} (\sigma(t_i) - t_i) + \lambda(E) \\
&= \sum_{i \in I_E} \lambda((t_i, \sigma(t_i))) + \lambda(E) \\
&= \lambda(\bigcup_{i \in I_E} (t_i, \sigma(t_i))) + \lambda(E) \\
&= \lambda(\bigcup_{i \in I_E} (t_i, \sigma(t_i)) \cup E) \\
&= \lambda(E^\sim). \qquad \square
\end{aligned}$$

Corollary 2.11 *If a set* E *is Lebesgue measurable, then* $E \cap \mathbb{T}$ *is* Δ*-measurable.*

Proof Suppose that $E \subset \mathbb{T}$. Then by Proposition 2.4, E is Δ-measurable if and only if E is Lebesgue measurable.

Now suppose that $E \cap (\mathbb{R} \backslash \mathbb{T}) \neq \emptyset$ i.e., E has some points that do not belong to $\mathbb{T}$. We know $\mathbb{T}$ is Lebesgue measurable and if E is also Lebesgue measurable, the intersection of two Lebesgue measurable sets is also Lebesgue measurable hence Δ-measurable, this ends the proof. $\square$

Now we give the concept of Lebesgue ∇-measure on $\mathbb{T}$ as Guseinov defined (Guseinov 2003). Let $\Im_2$ denote the family of intervals of $\mathbb{T}$ of the form $(a,b] = \{t \in \mathbb{T} : a < t \leq b\}$ with $a, b \in \mathbb{T}$ and $a \leq b$. Obviously, $(a,a]$ is the empty set. Let $m_2 : \Im_2 \rightarrow [0,\infty]$ be the set function on $\Im_2$ that assigns each interval $(a,b]$ its length i.e., $m_2 : b - a \rightarrow m_2((a,b]) = b - a$. Then m_2 is a countably additive measure on $\Im_2$. Similar to μ_Δ, an outer measure m_2^* on the family of all subsets of $\mathbb{T}$ is given as follows: Let D be any subset of $\mathbb{T}$. If there exists at least one finite or countable system of intervals $U_j \in \Im_2$ $(j = 1, 2, ...)$ such that $D \subset \bigcup_j U_j$, then $m_2^* = \inf \sum_j m_2(U_j)$. If there is no such covering of D, then $m_2^*(D) = \infty$. The family $\mathbf{M}(m_2^*)$ of all m_2^*-measurable (or ∇-measurable) subsets of $\mathbb{T}$ is a σ-algebra where a subset D is said to be m_2^*-measurable if

$$m_2^*(A) = m_2^*(A \cap D) + m_2^*(A \cap D^c) \tag{2.21}$$

holds for all $A \in \mathbb{T}$, $D^c = \mathbb{T} - D$. Then, we take the restriction of m_2^* to $\mathbf{M}(m_2^*)$, which is denoted by μ_∇. This μ_∇ is countably additive measure on $\mathbf{M}(m_2^*)$.

Theorem 2.12 *For each $t_0 \in \mathbb{T} - \{\min \mathbb{T}\}$, the ∇-measure of the single point set $\{t_0\}$ is given by*

$$\mu_\nabla(\{t_o\}) = t_0 - \rho(t_0). \tag{2.22}$$

Proof As in the proof of Theorem 2.2.

Theorem 2.13 *If $c, d \in \mathbb{T}$, then $\mu_\nabla((c,d]) = c - d$, $\mu_\nabla((c,d)) = \rho(d) - c$.*
If $c, d \in \mathbb{T} - \{\min \mathbb{T}\}$, then $\mu_\nabla([c,d)) = \rho(d) - \rho(c)$, $\mu_\nabla([c,d]) = d - \rho(c)$.

Proof As in the proof of Theorem 2.6.

Remark 2.14 *In case $\mathbb{T} = \mathbb{R}$, both measures coincide and equal to the usual Lebesgue measure. Another special case where these two measure coincide is the case $\mathbb{T} = h\mathbb{Z}$ where h is any positive constant.*

CHAPTER 3

Δ-MEASURABLE FUNCTIONS

3.1. Δ-Measurable Functions

Our aim is to go to details on Δ-measurable functions with properties, important theorems and results in order to understand the concept of Lebesgue Δ-integral.

In this chapter Δ-measurable functions are introduced specifically and comparison of Lebesgue measurable and Δ-measurable functions as Cabada-Vivero established take place in detail. Furthermore, we will introduce some properties of functions which expresses conditions for measurability of functions on time scales.

Definition 3.1 *We say that $f : \mathbb{T} \to \overline{\mathbb{R}}$ is Δ-measurable if for every $\alpha \in \mathbb{R}$, the set*

$$f^{-1}([-\infty, \alpha)) = \{t \in \mathbb{T} : f(t) < \alpha\} \tag{3.1}$$

is Δ-measurable.

To begin with, we will focus on simple function as given by Definition 1.9. However, in order to construct Lebesgue Δ-integral, we will be concerned about the case that S is Δ-measurable.

Remark 3.1 *It is easy to verify that the relationship between simple functions and Δ-measurable sets, that is, $S = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ is Δ-measurable if and only if each A_i is Δ-measurable set.*

Proof First suppose that the simple function $S : \mathbb{T} \to \overline{\mathbb{R}}$ is Δ-measurable. By Definition 3.1, the set

$$S^{-1}([-\infty, \alpha)) = \{t \in \mathbb{T} : S(t) < \alpha\} \tag{3.2}$$

is Δ-measurable for all $\alpha \in \mathbb{R}$ with $\alpha_1 < \alpha_2 < ... < \alpha_n$.

i)If $\alpha < \alpha_1$, then $S^{-1}([-\infty, \alpha)) = \emptyset$ which is known Δ-measurable.

ii) If $\alpha_1 < \alpha < \alpha_2$, then $S^{-1}([-\infty, \alpha)) = A_1$ is Δ-measurable.

iii) Let $\alpha_2 < \alpha < \alpha_3$. In this case, $S^{-1}([-\infty, \alpha)) = A_1 \cup A_2$ is Δ-measurable. From **ii)**, A_2 is Δ-measurable too since the difference of Δ-measurable sets is Δ-measurable.

Repeating this procedure, for any value of α, the corresponding set $\bigcup_{i=1}^{k} A_i$, $k = 1, 2, ..., n$ is Δ-measurable, we reach the result, each A_i is Δ-measurable.

Now, suppose that each A_i is Δ-measurable for $i = 1, 2, ...n$. Then for $0 \leq k \leq n$, the union $\bigcup_{i=1}^{k} A_i$ is also Δ-measurable which corresponds the set $\{t \in \mathbb{T} : S(t) < \alpha\}$ for some $\alpha \in \mathbb{R}$. For every $\alpha \in \mathbb{R}$ there exists union $\bigcup_{i=1}^{k} A_i$ we conclude that the corresponding simple function S is Δ-measurable. $\square$

The following proposition gives different versions of the definition of Δ-measurable function.

Proposition 3.1 *Let f be an extended real valued function whose domain is Δ-measurable set $E \subset \mathbb{T}$. Then the following statements are equivalent:*

i) *For each real number α, the set $\{t \in E : f(t) < \alpha\}$ is Δ-measurable.*

ii) *For each real number α, the set $\{t \in E : f(t) \geq \alpha\}$ is Δ-measurable.*

iii) *For each real number α, the set $\{t \in E : f(t) \leq \alpha\}$ is Δ-measurable.*

iv) *For each real number α, the set $\{t \in E : f(t) > \alpha\}$ is Δ-measurable.*

These statements imply

v) *For each real number α, the set $\{t \in E : f(t) = \alpha\}$ is Δ-measurable.*

Proof As in (Royden 1988).

Theorem 3.2 *Let f is Δ-measurable on $E \subset \mathbb{T}$ and $f = g$ Δ-a.e., then g is Δ-measurable too.*

Proof Considering the definition of Δ-measurability and equivalence Δ-a.e., the difference between sets $A = \{t : f(t) > \alpha\}$ and $B = \{t : g(t) > \alpha\}$ is nothing but the set with Δ-measure zero, that is,

$$\mu_\Delta((A \setminus B) \cup (B \setminus A)) = 0.$$

Thus,

$$\mu_\Delta(A \setminus B) = 0 \text{ and } \mu_\Delta(B \setminus A) = 0.$$

Without losing of generality let us assume f is Δ-measurable. Considering $B = A \cup (B \setminus A)$ where right hand side of the equation is Δ-measurable since it is the union of two Δ-measurable sets, this ends the proof. $\square$

Proposition 3.2 *Let f and g be real valued Δ-measurable functions defined on $E \subset \mathbb{T}$, α a real number. Then αf, $f+g$, $f-g$, fg, $\sup f_n$, $\inf f_n$, $\overline{\lim} f_n$, $\underline{\lim} f_n$ and $\dfrac{f}{g}$ are also Δ-measurable.*

Proof As in (Cohn 1997).

Corollary 3.3 *Let $\{f_n(t)\}$ be a sequence of Δ-measurable functions on $E \subset \mathbb{T}$. If this sequence is convergent with $\lim f_n(t) = f(t)$, then $f(t)$ is Δ-measurable too.*

Proof As in (Dönmez 2001).

Corollary 3.4 *Let $f_1(t)$ and $f_1(t)$ be two Δ-measurable functions defined on a Δ-measurable subset of $\mathbb{T}$. Then $\max\{f_1(t), f_2(t)\}$ and $\min\{f_1(t), f_2(t)\}$ are Δ-measurable.*

Proof As in (Dönmez 2001).

Theorem 3.5 *Every constant function defined on a Δ-measurable set E is Δ-measurable.*

Proof Let $f(t) = c$ for all $t \in E$, where $c \in \mathbb{R}$. For the set $\{t \in E : f(t) = c > \alpha\}$, it is convenient to consider α in two cases:

i) For $\alpha \geq c$ this set corresponds E which is known Δ-measurable.

ii) For $\alpha < c$ this set corresponds the empty set which is already known Δ-measurable. $\square$

Proposition 3.3 *If f_1 and f_2 are Δ-measurable functions, then the set $\{t : f_1(t) \leq f_2(t)\}$ is Δ-measurable.*

Proof As in (Dönmez 2001).

Corollary 3.6 *Now, it is easy to see in order that f is Δ-measurable, it is necessary and sufficient that both f^+ and f^- are Δ-measurable functions, where $f^+ = \max\{0, f\}$ and $f^- = \max\{0, -f\}$.*

Proof As in (Aliprantis and Burkinshaw 1998).

Theorem 3.7 *f is a Δ-measurable function if and only if for any $\alpha_1 < \alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$, the set $\{t \in \mathbb{T} : \alpha_1 < f(t) < \alpha_2\}$ is Δ-measurable.*

Proof As in (Craven 1982).

Remark 3.8 *For simplicity let us use the definition of Δ-measurable as follows: f is Δ-measurable if and only if for any open interval $I \subset \mathbb{T}$, $f^{-1}(I)$ is Δ-measurable.*

Using this fact, we will integrate two concepts: continuity and measurability.

Proposition 3.4 *Let f be a function defined on a totally discrete time scale $\mathbb{T}$. Then f is Δ-measurable.*

Proof It is known that a single point set so any countable union of single point sets is Δ-measurable. Hence $\mathbb{T}$ can be representded as a countable union of single point sets which are known Δ-measurable, for any function f defined on this time scale, $f^{-1}(I)$, that is, inverse image of any open interval under f can also be represented as some union of single point sets so f is Δ-measurable. □

This shows Δ-measurability of any function defined on a time scale which only has isolated points. We need to specify the time scale as it has just have isolated points, not dense-points, otherwise we are not able to generalize this fact to all possible functions that are defined on any other time scale because of the fact that we are not able to know the character of a function on right-dense points, whether inverse image of any open interval is Δ-measurable or not.

Remark 3.9 *It is easy to see that if f is defined on a subset of a totally discrete time scale $\mathbb{T}$, then f is Δ-measurable.*

Corollary 3.10 *Let f be defined on any time scale $\mathbb{T}$ and let $f|_{S_R}$ represent the function which is obtained by inducing f to the set of all right-scattered points of $\mathbb{T}$. Then $f|_{S_R}$ is Δ-measurable on S_R.*

Proposition 3.5 *If f is rd-continuous, then f is Δ-measurable.*

Proof In order to show Δ-measurability of f, we will use definition of Δ-measurable function given by Remark 3.8 and identity

$$f^{-1}(I) = f|_{D_R}^{-1}(I) \cup f|_{S_R}^{-1}(I) \tag{3.3}$$

where $f|_{S_R}$ represents the function which is obtained by inducing f to the set of all right-scattered points of $\mathbb{T}$ and $f|_{D_R}$ represents the function which is obtained by inducing

$f|_{S_R}$ to the set of all right-dense points of $\mathbb{T}$. From Corollary 3.10, $f|_{S_R}$ is Δ-measurable so we need to show $f|_{D_R}$ is a Δ-measurable function.
$f|_{D_R}$ is continuous so for a given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f|_{D_R}(s) - f|_{D_R}(t)| < \epsilon \text{ for all } s \in (t-\delta, t+\delta)$$

In another way

$$\begin{aligned}(t-\delta, t+\delta) &= \{s \in \mathbb{T} : |f|_{D_R}(s) - f|_{D_R}(t)| < \epsilon\} \\ &= \{s \in \mathbb{T} : f|_{D_R}(t) - \epsilon < f|_{D_R}(s) < f|_{D_R}(t) + \epsilon\}\end{aligned}$$

The left hand side of the equation is Δ-measurable so is the right hand side. By Remark 3.8, $f|_{D_R}$ is Δ-measurable, so it is obtained the right hand side of Equality 3.3 is Δ-measurable set, so is $f^{-1}(I)$ for any interval $I \subset \mathbb{T}$, thus f is Δ-measurable function. $\square$

Corollary 3.11 *Let f be a continuous function defined on $\mathbb{T}$. Then f is Δ-measurable.*

Remark 3.12 *It is easy to see that if an rd-continuous function f is defined on a Δ-measurable set $E \subset \mathbb{T}$, then f is Δ-measurable function.*

The next proposition is given with this restriction which generalizes the link between measurability and continuity.

Proposition 3.6 *Let f be defined on a Δ-measurable subset E of $\mathbb{T}$. Then f is Δ-measurable if the set of all right-dense points of E where f is discontinuous is a set of Δ-measure zero.*

Proof Let us classify E into three parts:

i) $C_{rd} = \{t \in E : f \text{ is continuous at } t \text{ and } t \text{ is right-dense}\}$

ii) $D_{rd} = \{t \in E : f \text{ is discontinuous at } t \text{ and } t \text{ is right-dense}\}$

iii) $E_{rs} = \{t \in E : t \text{ is right-scattered}\}$

Thus, $E = C_{rd} \cup D_{rd} \cup E_{rs}$. Let $\mu_\Delta(D_{rd}) = 0$. We will approach to D_{rd}, by defining a decreasing sequence of Δ-measurable sets. Let $\{K_n\}_{n\in\mathbb{N}}$ be the sequence of Δ-measurable sets such that

$$D_{rd} \subset K_n \subset (D_{rd} \cup C_{rd}) \text{ for all } n \in \mathbb{N}.$$

By defining $D_n = \bigcap_{r \leq n} K_r$, we obtain a decreasing sequence of sets $\{D_n\}_{n\in\mathbb{N}}$. Let $D_{rd} = \bigcap_{n=1}^{\infty} D_n$. It is obvious that f is continuous on F_n, where F_n is the complement of D_n with respect to $D_{rd} \cup C_{rd}$ i.e., $F_n = (D_{rd} \cup C_{rd}) \setminus D_n$. Hence the restriction $f|_{F_n}$ of f is continuous which implies that $f|_{F_n}$ is Δ-measurable from Corollary 3.11.

$\{F_n\}$ is an increasing sequence of sets as a result of $\{D_n\}$ is decreasing. Taking the limit as $n \to \infty$, $D_{rd} \cup C_{rd} = D_n \cup F_n$,

$$\begin{aligned}
D_{rd} \cup C_{rd} &= \lim_{n\to\infty} (D_n \cup F_n) \\
&= (\lim_{n\to\infty} D_n) \cup (\lim_{n\to\infty} F_n) \\
&= (\bigcap_{n=1}^{\infty} D_n) \cup (\bigcup_{n=1}^{\infty} F_n) \\
&= (D_{rd}) \cup (\bigcup_{n=1}^{\infty} F_n).
\end{aligned}$$

Using this fact, we will investigate Δ-measurability of the function f defined on $E = D_{rd} \cup (\bigcup_{n=1}^{\infty} F_n) \cup E_{rs}$ and similar to Equation 3.3,

$$\begin{aligned}
f^{-1}(I) &= (f|_{D_{rd}}^{-1}(I)) \cup (f|_{\bigcup_{n=1}^{\infty} F_n}^{-1}(I)) \cup (f|_{E_{rs}}^{-1}(I)) \\
&= (f|_{D_{rd}}^{-1}(I)) \cup (\bigcup_{n=1}^{\infty} f|_{F_n}^{-1}(I)) \cup (f|_{E_{rs}}^{-1}(I)).
\end{aligned}$$

The right hand side of the equation is Δ-measurable since it is a countable union of Δ-measurable sets, so is the left hand side, this ends the proof. $\square$

Theorem 3.13 *Let $f : \mathbb{T} \to \mathbb{R}^+ \cup \{0\}$ be a Δ-measurable function. There exists a sequence $\{\varphi_n\}$ of Δ-measurable simple functions such that $0 \leq \varphi_n(t) \uparrow f(t)$ holds for all $t \in \mathbb{T}$.*

Proof Let $f : \mathbb{T} \to \mathbb{R}^+ \cup \{0\}$ be a Δ-measurable function and let

$$A_n^i = \{t \in \mathbb{T} : (i-1)2^{-n} \leq f(t) < i2^{-n}\} \text{ for } i = 1, 2, ...n2^n$$

Note that $A_n^i \cap A_n^j = \emptyset$ if $i \neq j$. Since f is a Δ-measurable function, then each A_i is Δ-measurable set. For each n we define a sequence of Δ-measurable simple functions $\{\varphi_n\}$ as $\varphi_n = \sum_{i=1}^{n2^n} 2^{-n}(i-1)\chi_{A_n^i}$ such that $0 \leq \varphi_n(t) \leq \varphi_{n+1}(t) \leq f(t)$ holds for all $t \in \mathbb{T}$ and all $n \in \mathbb{N}$.

Furthermore, if t is fixed, then $0 \leq f(t) - \varphi_n(t) \leq 2^{-n}$ holds for all sufficiently large n. Thus, $\varphi_n \uparrow f(t)$ holds for all t. (Rudin 1987) $\square$

Theorem 3.14 *f is Δ-measurable if and only if it can be represented as the limit of a sequence of Δ-measurable simple functions which is uniformly convergent.*

Proof If $f(t)$ is the limit of a uniformly convergent sequence of Δ-measurable functions, then $f(t)$ is Δ-measurable from the previous theorem.

Conversely, let us consider any Δ-measurable function $f(t)$ and let m and n are two positive integers. If

$$\frac{m}{n} \leq f(t) < \frac{m+1}{n} \text{ then take } f_n(t) = \frac{m}{n}$$

Since $\frac{m}{n}$ is rational number and $f_n(t)$ takes at most countable values and Δ-measurable, hence $f_n(t)$s are each simple functions.

$$|f(t) - f_n(t)| \leq |\frac{m+1}{n} - \frac{m}{n}| = \frac{1}{n}$$

As n tends to infinity, $f_n(t)$ approaches to $f(t)$ which gives the definition of uniformly convergence for $\{f_n(t)\}_{n\in\mathbb{N}}$. (Dönmez 2001) $\square$

3.2. Lebesgue Measurable and Lebesgue Δ-Measurable Functions

In order to compare measurable functions and Δ-measurable functions, we need to define a function f on $\mathbb{T}$, extend it and denote the extension as $f^{\sim}$ to the interval $\mathbb{T}^{\sim}$ as follows:

$$f^{\sim}(t) = \begin{cases} f(t) & if\ t \in \mathbb{T} \\ f(t_i) & if\ t \in (t_i, \sigma(t_i)) \end{cases} \tag{3.4}$$

for some $i \in I$, where $\{t_i\}_{i\in I}$ represents the set of all right-scattered points in $\mathbb{T}$ and $\mathbb{T}^{\sim}$ is the extension of $\mathbb{T}$ as given in Remark 2.4.

Proposition 3.7 *Let f be defined from $\mathbb{T}$ to the extended real line and $f^{\sim}$ is the extension of f introduced in Equation 3.4. Then f is Δ-measurable function if and only if $f^{\sim}$ is Lebesgue measurable function.(Cabada and Vivero 2004)*

Proof Suppose that f is Δ-measurable i.e., for any real number α, $f^{-1}([-\infty, \alpha)) = A_\alpha$ is a Δ-measurable set. We need to prove for each α, $f^{\sim -1}([-\infty, \alpha))$ is Lebesgue measurable.

$$\begin{aligned}
f^{\sim -1}([-\infty,\alpha)) &= \{t \in \mathbb{T} \cup \bigcup_{i \in I_{\mathbb{T}}} (t_i, \sigma(t_i)) : f^{\sim}(t) < \alpha\} \\
&= \{t \in \mathbb{T} : f^{\sim}(t) < \alpha\} \cup \{t \in \bigcup_{i \in I_{\mathbb{T}}} (t_i, \sigma(t_i)) : f^{\sim}(t) < \alpha\} \\
&= \{t \in \mathbb{T} : f(t) < \alpha\} \cup \{t \in \bigcup_{i \in I_{\mathbb{T}}} (t_i, \sigma(t_i)) : f^{\sim}(t) < \alpha\} \\
&= \{t \in \mathbb{T} : f(t) < \alpha\} \cup \big(\bigcup_{i \in I_{A_\alpha}} (t_i, \sigma(t_i))\big) \\
&= A_\alpha \cup \big(\bigcup_{I_{A_\alpha}} (t_i, \sigma(t_i))\big) = A_\alpha^{\sim}.
\end{aligned}$$

Since f is Δ-measurable function from the assumption, A_α is a Δ-measurable set, furthermore from Proposition 2.4, it is Lebesgue measurable. Hence $f^{\sim -1}([-\infty,\alpha)) = A_\alpha^{\sim}$ is Lebesgue measurable since it is represented as countable union of Lebesgue measurable sets.

Now suppose that $f^{\sim -1}$ is Lebesgue measurable. Using the identity:

$$f^{\sim -1}([-\infty,\alpha)) \cap \mathbb{T} = f^{-1}([-\infty,\alpha)),$$

we conclude that $\mathbb{T}$ and $f^{\sim -1}([-\infty,\alpha))$ is Lebesgue measurable, then $f^{-1}([-\infty,\alpha))$ is Δ-measurable from Corollary 2.11. $\square$

In this way we can easily see the relation between the usual Lebesgue integral and Lebesgue- Δ integral which we will consider in the next chapter.

CHAPTER 4

Δ-INTEGRAL

By definition, a Riemann integrable function is bounded and its domain is a closed interval. These two restriction make the Riemann integral inadequate to fulfill the requirements of many scientific problems. H. Lebesgue introduced a concept of an integral based on measure theory that generalizes the Riemann integral. It has the advantage that at the same time it treats both bounded and unbounded functions and allows their domains to be more general sets. Also, it gives more powerful and useful convergence theorems than the Riemann integral does.(Aliprantis and Burkinshaw 1998)

In this chapter, we will give the general concept of Riemann Δ-integral was introduced by Guseinov and Kaymakcalan (Guseinov and Kaymakcalan, 2002), by defining Darboux Δ-sum and Darboux Δ-integral first without going to details just giving basics in order to compare and integrate with Lebesgue Δ-integral and show Riemann Δ-integral is a special form for Lebsegue Δ-integral. Further, we will introduce Lebesgue Δ-integral considering firstly Δ-measurable simple functions' integrals. Finally comparison of both Δ-integral and Lebesgue integral of Δ-measurable functions follow taking into account Cabada-Vivero's related paper (Cabada and Vivero 2004).

4.1. Riemann Δ-Integral

Let $\mathbb{T}$ be a time scale, $[a, b)$ be a half closed interval in $\mathbb{T}$, a partition of $[a, b)$ is any ordered subset $P = \{t_0, t_1, ...t_n\} \subset [a, b]$ where $a = t_0 < t_1 < ... < t_n = b$.
Let f be a real valued bounded function defined on $[a, b) \subset \mathbb{T}$.
Here we set

$$M = \sup\{f(t) : t \in [a, b)\}, \qquad m = \inf\{f(t) : t \in [a, b)\},$$
$$M_i = \sup\{f(t) : t \in [t_{i-1}, t_i)\}, \quad m_i = \inf\{f(t) : t \in [t_{i-1}, t_i)\}.$$

The upper Darboux Δ-sum $U(f, P)$ and the lower Darboux Δ-sum $L(f, P)$ of the function f are defined by

$$U(f, P) = \sum_{n=1}^{n} M_i(t_i - t_{i-1}), \quad L(f, P) = \sum_{n=1}^{n} m_i(t_i - t_{i-1}) \tag{4.1}$$

with respect to the partition P. The upper Darboux integral $U(f)$ of f from a to b is defined by

$$U(f) = \inf\{U(f, P)\}, \tag{4.2}$$

and the lower Darboux integral $L(f)$ of f from a to b is defined by

$$L(f) = \sup\{U(f, P)\}. \tag{4.3}$$

Definition 4.1 *f is said to be Δ-integrable from a to b if $L(f) = U(f)$. In this case, we write $\int_a^b f(t)\Delta t$ for this common value and call this integral, the Darboux Δ-integral.*

Lemma 4.1 *For every $\delta > 0$ there exists at least one partition $P : a = t_0 < t_1 < ... < t_n = b$ of $[a, b)$ such that for each $i \in \{1, 2, ...n\}$ either $t_i - t_{i-1} \leq \delta$ or $t_i - t_{i-1} > \delta$ and $\rho(t_i) = t_{i-1}$, where ρ denotes the backward jump operator in $\mathbb{T}$.(Guseinov 2003)*

Proof (Guseinov 2003).

Theorem 4.2 *A bounded function f on $[a, b)$ is Δ-integrable if and only if for each $\epsilon > 0$, there exists $\delta > 0$ such that*

$$U(f, P) - L(f, P) < \epsilon. \tag{4.4}$$

for all partitions $P \in \wp_\delta$ where $\wp_\delta$ denotes the set of all partitions P that posses the property indicated in Lemma 4.33.(Guseinov 2003)

Proof (Guseinov 2003).

We now give the concept of Riemann Δ-integral.

Definition 4.2 *Let f be a bounded function on $[a, b)$ and let $P : a = t_0 < t_1 < ... < t_n = b$ be a partition of $[a, b)$. In each interval $[t_{i-1}, t_i)$, where $i = 1, 2, ...n$, we choose an arbitrary point ξ_i and form the sum*

$$S = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}). \tag{4.5}$$

We call S a Riemann Δ-sum of f corresponding to the partition P.

We say that f is Riemann Δ-integrable form a to b if there exists a number I with the following property: For each $\epsilon > 0$, there exists $\delta > 0$ such that $|S - I| < \epsilon$ for every Riemann Δ-sum S of f corresponding to a partition $P \in \wp_\delta$, with the property that was given in Lemma 4.33, independent of the way in which we choose $\xi_i \in [t_{i-1}, t_i)$, $i = 1, 2, ...n$. It is easy to see that such a number I is unique. The number I is the Riemann Δ-integral of f from a to b.

Theorem 4.3 *A bounded function f on $[a,b)$ is Riemann Δ-integrable if and only if it is (Darboux) Δ-integrable, in which case the values of the integrals are equal.*

Proof (Guseinov 2003).

4.2. Lebesgue Δ-Integral

Definition 4.3 *Let $E \subset \mathbb{T}$ be a Δ-measurable set and let $S : \mathbb{T} \to [0,+\infty)$ be a Δ-measurable simple function with $S = \sum_{i=1}^{n} \alpha_i \chi_{A_j}$, $A_j = \{t : S(t) = \alpha_i\}$.*
The Lebesgue Δ-integral of S on E is defined as

$$\int_E S(s)\Delta s = \sum_{i=1}^{n} \alpha_i \mu_\Delta(A_i \cap E). \tag{4.6}$$

Proposition 4.1 *Let $\{E_n\}_{n\in\mathbb{N}}$ be an increasing sequence of Δ-measurable sets and let S be a nonnegative Δ-measurable simple function on $E = \lim_{j\to\infty} E_j$. Then*

$$\int_E S(s)\Delta s = \lim_{j\to\infty} \int_{E_j} S(s)\Delta s. \tag{4.7}$$

Proof Let S take value α_i on each Δ-measurable set A_i. Then

$$\int_{E_j} S(s)\Delta s = \sum_{i=1}^{n} \alpha_i \mu_\Delta(E_j \cap A_i) \tag{4.8}$$

and taking the limit of both sides as $j \to \infty$,

$$\begin{aligned}
\lim_{j\to\infty} \int_{E_j} S(s)\Delta s &= \lim_{j\to\infty} \sum_{i=1}^{n} \alpha_i \mu_\Delta(E_j \cap A_i) \\
&= \sum_{i=1}^{n} \alpha_i \lim_{j\to\infty} \mu_\Delta(E_j \cap A_i) \\
&= \sum_{i=1}^{n} \alpha_i \mu_\Delta\big(\bigcup_{j=1}^{\infty}(E_j \cap A_i)\big) \\
&= \sum_{i=1}^{n} \alpha_i \mu_\Delta\big((\bigcup_{j=1}^{\infty} E_j) \cap A_i\big) \\
&= \sum_{i=1}^{n} \alpha_i \mu_\Delta(E \cap A_i) \\
&= \int_E S(s)\Delta s \qquad \square
\end{aligned}$$

Lemma 4.4 *Let $E \subset \mathbb{T} - \{\max \mathbb{T}\}$ be a Δ-measurable set, $S : \mathbb{T} \to [0, \infty)$ be a Δ-measurable simple function with $S = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ and $S^{\sim}$ be the extension of S given by Equation 3.4, of the form $S^{\sim} = \sum_{i=1}^{n} \alpha_i \chi_{A_i^{\sim}}$ where $A_i^{\sim} = A_i \cup (\bigcup_{i \in I_{A_i}} (t_i, \sigma(t_i)))$. Then $S^{\sim}$ is Lebesgue measurable and Lebesgue integrable with*

$$\int_E S(s)\Delta s = \int_{E^{\sim}} S^{\sim}(s)ds \tag{4.9}$$

where $E^{\sim}$ is the extension of E as given in Remark 2.4.(Cabada and Vivero 2004)

Proof Measurability of $S^{\sim}(t)$ is obvious from the fact that extension of Δ-measurable function is Lebesgue measurable. Furthermore, $E^{\sim} \cap A_i^{\sim} = (E \cap A_i)^{\sim}$. Since $S^{\sim}$ is Lebesgue measurable simple function on $E^{\sim}$, then it is Lebesgue integrable with

$$\begin{aligned}
\int_{E^{\sim}} S^{\sim}(s)ds &= \sum_{i=1}^{n} \alpha_i \lambda(E^{\sim} \cap A_i^{\sim}) \\
&= \sum_{i=1}^{n} \alpha_i \lambda(E \cap A_i)^{\sim} \\
&= \sum_{i=1}^{n} \alpha_i \mu_{\Delta}(E \cap A_i) \\
&= \int_E S(s)\Delta s. \qquad \square
\end{aligned}$$

Definition 4.4 *Let $E \subset \mathbb{T}$ be a Δ-measurable set and let $f : \mathbb{T} \to [0, +\infty]$ be a Δ-measurable function. The Lebesgue Δ-integral of f on E is defined as*

$$\int_E f(s)\Delta s = \sup \int_E S(s)\Delta s$$

where the supremum is taken over all Δ-measurable nonnegative simple functions S such that $S \leq f$ defined on $\mathbb{T}$.

Theorem 4.5 ***(Beppo Levi's Lemma Adapted to Time Scale)*** *Assume that a sequence $\{f_n\}_{n \in \mathbb{N}}$ of Δ-integrable functions on a Δ-measurable set E satisfies $f_n(t) \leq f_{n+1}(t)$ Δ-a.e., for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \int_E f_n(s)\Delta s < \infty$. Then there exists a Δ-integrable function f such that $f_n \uparrow f$ and hence $\int_E f_n(s)\Delta s \uparrow \int_E f(s)\Delta s$.*

Proof As in (Aliprantis and Burkinshaw 1998).

Theorem 4.6 ***(Fatou's Lemma Adapted to Time Scale)*** *Let $\{f_n(t)\}_{n\in\mathbb{N}}$ be a sequence of Δ-integrable functions defined on a Δ-measurable set E such that $f_n(t) \geq 0$ Δ-a.e. and $\underline{\lim} \int_E f_n(s)\Delta s < \infty$. Then, $\underline{\lim} f_n(t)$ defines a Δ-integrable function on the Δ-measurable set E and*

$$\int_E \underline{\lim} f_n(s)\Delta s \leq \underline{\lim} \int_E f_n(s)\Delta s. \tag{4.10}$$

Proof Without loss of generality, we suppose that $f_n(t) \geq 0$ holds for all $t \in E$. We define a sequence $\{g_n\}_{n\in\mathbb{N}}$ such that $g_n(t) = \inf\{f_i(t) : i \geq n\}$ for each $t \in \mathbb{N}$. Since each $f_i(t)$, $i \geq n$ is Δ-measurable, then $g_n(t)$s are Δ-measurable and Δ-integrable with $0 \leq g_n(t) \leq f_n(t)$ holds for all $n \in \mathbb{N}$. By Theorem 4.5, there exists a Δ-measurable function $g(t)$ such that $g_n(t) \uparrow g(t)$ Δ-a.e.
It follows that $g(t) = \underline{\lim} f_n(t)$ and $\int_E g(t) = \int_E \underline{\lim} f_n(t)$.

$$\int_E \underline{\lim} f_n(s)\Delta s = \int_E g(s)\Delta s = \lim \int_E g_n(s)\Delta s \leq \underline{\lim} \int_E f_n(s)\Delta s \tag{4.11}$$

Whence

$$\int_E \underline{\lim} f_n(s)\Delta s \leq \underline{\lim} \int_E f_n(s)\Delta s \qquad \Box \tag{4.12}$$

Theorem 4.7 ***(Monotone Convergence Theorem Adapted to Time Scale)*** *Let $\{f_n\}_{n\in\mathbb{N}}$ be an increasing sequence of nonnegative Δ-measurable functions defined on a Δ-measurable set E and let $f(t) = \lim f_n(t)$ Δ-a.e.*
Then

$$\int_E \lim f(s)\Delta s = \lim \int_E f_n(s)\Delta s. \tag{4.13}$$

Proof As in (Aliprantis and Burkinshaw 1998).

Theorem 4.8 ***(Lebesgue Dominated Convergence Theorem Adapted to Time Scale)*** *Let g be Δ-integrable function over E and let $\{f_n\}$ be sequence of Δ-measurable functions such that*

$$| f_n(t) | \leq g(t)$$

on E and for almost all t in E we have $f(t) = \lim f_n(t)$. Then

$$\int_E f(t)\Delta t = \lim \int_E f_n(t)\Delta t.$$

Proof As in (Aliprantis and Burkinshaw 1998).

Remark 4.9 *Let $\{f_n\}_{n\in\mathbb{N}}$ be an increasing (or decreasing) sequence of functions defined on $\mathbb{T}$ and let $f_n^{\sim}(t)$ is the extension of f_n as defined previously. Then $\{f_n^{\sim}\}_{n\in\mathbb{N}}$ is an increasing (or decreasing) sequence of functions defined on the extension of $\mathbb{T}^{\sim}$, which is defined in Remark 2.4.*

The next lemma exhibits the relation between the classical Lebesgue integral and Lebesgue Δ-integral of nonnegative Δ-measurable functions.

Lemma 4.10 *Let $E \subset \mathbb{T}$ be a Δ-measurable set and do not include the maximum point of $\mathbb{T}$, if $\mathbb{T}$ is bounded from above, let $f : \mathbb{T} \to [0, +\infty]$ be a Δ-measurable function and $f^{\sim}$ be the extension of f as Equation 3.4. Then*

$$\int_E f(s)\Delta s = \int_{E^{\sim}} f^{\sim}(s)ds \tag{4.14}$$

where $E^{\sim} = E \cup (\bigcup_{I_E} (t_i, \sigma(t_i)))$. (Cabada and Vivero 2004)

Proof We will approach to $f(t)$ by a sequence of nonnegative Δ-measurable simple functions $\{S_n\}_{n\in\mathbb{N}}$ such that for all $t \in \mathbb{T}$

i) $0 \le S_n(t) \le S_{n+1}(t) \le f(t)$ and

ii) $\lim_{n\to\infty} S_n(t) = f(t)$.

We can extend the elements of this sequence as

i) $0 \le S_n^{\sim}(t) \le S_{n+1}^{\sim}(t) \le f^{\sim}(t)$ and

ii) $\lim_{n\to\infty} S_n^{\sim}(t) = f^{\sim}(t)$ since monotonicity stays the same.

By using Theorem 4.7,

$$\begin{aligned} \int_E f(s)\Delta s &= \lim_{n\to\infty} \int_E S_n(s)\Delta s \\ &= \lim_{n\to\infty} \int_{E^{\sim}} S_n^{\sim}(s)ds \\ &= \int_{E^{\sim}} f^{\sim}(s)ds. \qquad \square \end{aligned}$$

Definition 4.5 *Let $E \subset \mathbb{T}$ be a Δ-measurable set and let $f : \mathbb{T} \to \overline{\mathbb{R}}$ be a Δ-measurable function. We say that f is Lebesgue Δ-integrable on E if at least one of the elements*

$\int_E f^+(s)\Delta s$ or $\int_E f^-(s)\Delta s$ is finite, where f^+ and f^- are the positive and negative part of f respectively. In this case, we define the Lebesgue Δ-integral of f on E as

$$\int_E f(s)\Delta s = \int_E f^+(s)\Delta s - \int_E f^-(s)\Delta s. \tag{4.15}$$

Proposition 4.2 *Let let f and g be real-valued Δ-integrable functions on a Δ-measurable set $E \subset \mathbb{T}$ and let a be a real number. Then*

a) *af and $f+g$ are Δ-integrable on E.*

b) *$\int_E af(s)\Delta s = a\int_E f(s)\Delta s$.*

c) *$\int_E (f(s)+g(s))\Delta s = \int_E f(s)\Delta s + \int_E f(s)\Delta s$.*

d) *If $f(t) \leq g(t)$ holds for each $t \in E$ then,*

$\int_E f(s)\Delta s \leq \int_E g(s)\Delta s$.

Proof As in (Royden 1988).

Remark 4.11 *Note that Lebesgue integral has the same value on $I = [a,b]$, $(a,b]$, $[a,b)$, or (a,b). However, the integral with respect to μ_Δ, differs depending on endpoints.*

Suppose that f is Δ-integrable on $[a,b)$.

i)

$$\begin{aligned} \int_{[a,b]} f(s)\Delta s &= \int_{[a,b)\cup\{b\}} f(s)\Delta s \\ &= \int_{[a,b)} f(s)\Delta s + \int_{\{b\}} f(s)\Delta s \\ &= \int_{[a,b)} f(s)\Delta s + f(b)\mu(b). \end{aligned}$$

ii)

$$\begin{aligned} \int_{[a,b)} f(s)\Delta s &= \int_{\{a\}\cup(a,b)} f(s)\Delta s \\ &= \int_{\{a\}} f(s)\Delta s + \int_{(a,b)} f(s)\Delta s \\ &= f(a)\mu(a) + \int_{(a,b)} f(s)\Delta s \end{aligned}$$

Thus

$$\int_{(a,b)} f(s)\Delta s = \int_{[a,b)} f(s)\Delta s - f(a)\mu(a).$$

iii)

$$\begin{aligned}\int_{(a,b]} f(s)\Delta s &= \int_{(a,b)\cup\{b\}} f(s)\Delta s \\ &= \int_{(a,b)} f(s)\Delta s + \int_{\{b\}} f(s)\Delta s \\ &= \int_{[a,b)} f(s)\Delta s - f(a)\mu(a) + f(b)\mu(b).\end{aligned}$$

Theorem 4.12 *Let $E \subset \mathbb{T} - \{\max \mathbb{T}\}$ be a Δ-measurable set, $f : \mathbb{T} \to \overline{\mathbb{R}}$ be a Δ-measurable function and $f^{\sim}$ be the extension of f as given in Equation 3.4. Then, f is Lebesgue Δ-integrable on E if and only if $f^{\sim}$ is $Lebesgue$ integrable on $E^{\sim}$ where $E^{\sim}$ is defined by Remark 2.4. In that case, the following equality holds:*

$$\int_E f(s)\Delta s = \int_{E^{\sim}} f^{\sim}(s)ds. \tag{4.16}$$

(Cabada and Vivero 2004)

Proof We begin with considering the equality $\int_E f(s)\Delta s = \int_E f^{+}(s)\Delta s - \int_E f^{-}(s)\Delta s$. It is obvious that $(f^{+})^{\sim} = (f^{\sim})^{+}$ and similarly $(f^{-})^{\sim} = (f^{\sim})^{-}$. Substituting these identities

$$\begin{aligned}\int_E f(s)\Delta s &= \int_E f^{+}(s)\Delta s - \int_E f^{-}(s)\Delta s \\ &= \int_{E^{\sim}} (f^{+})^{\sim}(s)ds - \int_{E^{\sim}} (f^{-})^{\sim}(s)ds \\ &= \int_{E^{\sim}} (f^{\sim})^{+}(s)ds - \int_{E^{\sim}} (f^{\sim})^{-}(s)ds \\ &= \int_{E^{\sim}} f^{\sim}(s)ds. \qquad \square\end{aligned}$$

Theorem 4.13 *Let $E \subset \mathbb{T} - \{\max \mathbb{T}\}$ be a Δ-measurable set. If $f : \mathbb{T} \to \overline{\mathbb{R}}$ is Δ-integrable on E, then*

$$\int_E f(s)\Delta s = \int_E f(s)ds + \sum_{i\in I_E} f(t_i)\mu(t_i). \tag{4.17}$$

Proof We have

$$\int_E f(s)\Delta s = \int_{E^\sim} f^\sim(s)ds \;\text{ and }\; E^\sim = E \cup (\bigcup_{i\in I_E}(t_i,\sigma(t_i))).$$

Using these identities

$$\begin{aligned}
\int_E f(s)\Delta s = \int_{E^\sim} f^\sim(s)ds &= \int_{E\cup\bigcup_{i\in I_E}(t_i,\sigma(t_i))} f^\sim(s)ds \\
&= \int_E f^\sim(s)ds + \int_{\bigcup_{i\in I_E}(t_i,\sigma(t_i))} f^\sim(s)ds \\
&= \int_E f(s)ds + \sum_{i\in I_E}\int_{(t_i,\sigma(t_i))} f(t_i)ds \\
&= \int_E f(s)ds + \sum_{i\in I_E} f(t_i)\mu(t_i). \qquad \square
\end{aligned}$$

Theorem 4.14 ***(Approximation of Δ-Integral)*** *Let f be an rd-continuous function on an interval $I \subset \mathbb{T}$ and let $\{I_n\}$ be a sequence of intervals such that for all $I_n \subset I$, $\bigcap_{n=1}^{\infty} I_n = \{a\}$. Then*

$$\lim_{n\to\infty}\frac{1}{\mu_\Delta(I_n)}\int_{I_n} f(s)\Delta s = f(a). \tag{4.18}$$

Proof We will use the fact that $\bigcap_{n=1}^{\infty} I_n = \{a\}$ implies $\{\mu_\Delta(I_n)\} \to \mu(a)$, where μ denotes the graininess function. We need to consider in two cases: where a is a right-dense point and where a is a right-scattered point.

Suppose first a is a right-scattered point, then 4.18 is obvious from the definition of integral.

Suppose now a is a right-dense point. For this case, f is continuous. By continuity, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|t-a| < \delta \Rightarrow |f(t) - f(a)| < \epsilon$$

Then

$$f(a) - \epsilon < f(t) < f(a) + \epsilon.$$

Integrating over any interval $I_n \subset [a-\delta, a+\delta]$

$$\begin{aligned}
&\int_{I_n}(f(a)-\epsilon)\Delta s < \int_{I_n} f(s)\Delta s < \int_{I_n}(f(a)+\epsilon)\Delta s \\
&(f(a)-\epsilon)\int_{I_n}\Delta s < \int_{I_n} f(s)\Delta s < (f(a)+\epsilon)\int_{I_n}\Delta s \\
&(f(a)-\epsilon)\mu_\Delta(I_n) < \int_{I_n} f(s)\Delta s < (f(a)+\epsilon)\mu_\Delta(I_n) \\
&f(a)-\epsilon < \frac{1}{\mu_\Delta(I_n)}\int_{I_n} f(s)\Delta s < f(a)+\epsilon
\end{aligned}$$

$$-\epsilon < \frac{1}{\mu_\Delta(I_n)} \int_{I_n} f(s)\Delta s - f(a) < \epsilon$$
$$|\frac{1}{\mu_\Delta(I_n)} \int_{I_n} f(s)\Delta s - f(a)| < \epsilon.$$

Whence

$$\lim_{n\to\infty} \frac{1}{\mu_\Delta(I_n)} \int_{I_n} f(s)\Delta s = f(a). \qquad \square$$

Remark 4.15 *By using the definition of differentiation for right-dense points,*

$$\begin{aligned} \frac{d}{dt}\int_{[a,t)} f(s)\Delta s &= \lim_{c\to\infty} \frac{\int_{[a,t+c)} f(s)\Delta s - \int_{[a,t)} f(s)\Delta s}{c} \\ &= \lim_{n\to 0} \frac{1}{c} \int_{[t,t+c)} f(s)\Delta s \\ &= \lim_{n\to\ 0} \frac{1}{\mu_\Delta([t,t+c))} \int_{[t,t+c)} f(s)\Delta s \\ &= f(t). \end{aligned}$$

By using the definition of differentiation for right-scattered points,

$$\begin{aligned} \frac{d}{dt}\int_{[a,t)} f(s)\Delta s &= \frac{\int_{[a,\sigma(t))} f(s)\Delta s - \int_{[a,t)} f(s)\Delta s}{\mu(t)} \\ &= \frac{\int_{[t,\sigma(t))} f(s)\Delta s}{\mu(t)} \\ &= \frac{f(t)\mu(t)}{\mu(t)} \\ &= f(t). \end{aligned}$$

As a result, provided that f is rd-continuous and bounded function on $[a,b)$, the classical method of evaluating $\int f$ in terms of an antiderivative is applicable in the measure theory on time scale:

$$\int_{[a,b)} f(s)\Delta s = F(b) - F(a)$$

where F is antiderivative of f, $F'(t) = f(t)$ for each $t \in [a,b)$.

4.3. Relation Between Lebesgue Δ-Integral and Riemann Δ-Integral

Here is a comparison of the Lebesgue Δ-integral with the Riemann Δ-integral which was established by Guseinov, (Guseinov 2003).

Theorem 4.16 *Let $[a,b)$ be a half closed bounded interval in $\mathbb{T}$ and f be a bounded real-valued function on $[a,b)$. If f is Riemann Δ-integrable from a to b, then f is Lebesgue*

Δ-integrable on $[a,b)$ and $\int_a^b f(s)\Delta s = \int_{[a,b)} f(s)\Delta s$, where the left hand side of the equation is the Riemann Δ-integral and the right hand side of the equation is the Lebesgue Δ-integral.

Proof Suppose that f is Riemann Δ-integrable from a to b. Then, for each positive integer k we can choose $\delta_k > 0$, ($\delta_k \to 0$ as $k \to \infty$) and a partition $P_k : a = t_0^{(k)} < t_1^{(k)} < ...t_{n(k)}^{(k)} = b$ of $[a,b)$ such that $P_k \in \wp_{\delta_k}$ and $U(f,P_k) - L(f,P_k) < 1/k$. Hence

$$\lim_{k\to\infty} L(f,P_k) = \lim_{k\to\infty} U(f,P_k) = \int_a^b f(s)\Delta s. \tag{4.19}$$

By replacing the partitions P_k with finer partitions if necessary, we can assume that for each k partition P_{k+1} is a refinement of the partition P_k. Let us set

$$m_i^{(k)} = \inf\{f(t) : t \in [t_{i-1}^{(k)}, t_i^{(k)})\}, \quad M_i^{(k)} = \sup\{f(t) : t \in [t_{i-1}^{(k)}, t_i^{(k)})\} \tag{4.20}$$

for $i = 1,2,...,n(k)$ and define sequences (φ_k) and (ϕ_k) of functions on $[a,b)$ as

$$\varphi_k(t) = m_i^{(k)} \quad \text{and} \quad \phi_k(t) = M_i^{(k)} \quad for \quad t \in [t_{i-1}^{(k)}, t_i^{(k)}), \quad i = 1,2,...,n(k). \tag{4.21}$$

Then (φ_k) is a non-decreasing and (ϕ_k) is a non-increasing sequences of simple Δ-measurable functions with

$$\varphi_k \le \varphi_{k+1}, \quad \phi_k \ge \phi_{k+1}, \quad \varphi_k \le f \le \phi_k \tag{4.22}$$

$$\int_{[a,b)} \varphi_k(s)\Delta s = L(f,P_k), \quad \int_{[a,b)} \phi_k(s)\Delta s = U(f,P_k) \tag{4.23}$$

Since f is bounded, the sequences (φ_k) and (ϕ_k) are bounded by 4.21 and 4.20. Taking into account, the monotonicity 4.22 we conclude that there exist the limit functions

$$\varphi(t) = \lim_{k\to\infty} \varphi_k(t), \quad \phi(t) = \lim_{k\to\infty} \phi_k(t), \quad t \in [a,b). \tag{4.24}$$

It follows from 4.22 that

$$\varphi(t) \le f(t) \le \phi(t), \quad t \in [a,b). \tag{4.25}$$

The functions φ and ϕ are Δ-measurable as limits 4.24 of the Δ-measurable functions φ_k and ϕ_k, respectively. Lebesgue's dominated convergence theorem (see Theorem 4.8) implies that φ and ϕ are Lebesgue Δ-integrable on$[a,b)$ and

$$\lim_{k\to\infty} \int_{[a,b)} \varphi_k(s)\Delta s = \int_{[a,b)} \varphi(s)\Delta s \quad \text{and} \quad \lim_{k\to\infty} \int_{[a,b)} \phi_k(s)\Delta s = \int_{[a,b)} \phi(s)\Delta s \tag{4.26}$$

Therefore passing on to the limit in 4.23 as $k \to \infty$, we get

$$\int_{[a,b)} \varphi(s)\Delta s = \int_{[a,b)} \phi(s)\Delta s = \int_a^b f(s)\Delta s \tag{4.27}$$

and we have

$$\int_{[a,b)} [\phi(s) - \varphi(s)]\Delta s = 0. \tag{4.28}$$

Since in addition $\phi(s) - \varphi(s) \geq 0$ for all s in $[a,b)$, 4.28 implies $\varphi(s) = \phi(s)$ for Δ-almost every t in $[a,b)$ So $f(t) = \varphi(t)$ Δ-a.e. on $[a,b)$. It follows that f is Lebesgue Δ-integrable, and the Lebesgue Δ-integral of f over $[a,b)$ coincides with the Riemann Δ-integral of f from a to b. $\square$

The next theorem gives a Lebesgue Criterion for Riemann Δ-integrability.

Theorem 4.17 *Let f be a bounded function defined on the half-closed bounded interval $[a,b)$ of $\mathbb{T}$. Then f is Riemann Δ-integrable from a to b if and only if the set of all right-dense points of $[a,b)$ at which f is discontinuous is a set of Δ-measure zero. (Guseinov 2003)*

Proof Suppose that f is Riemann Δ-integrable from a to b. For each positive integer k, let a partition P_k of $[a,b)$ and the functions φ_k, ϕ_k, φ and ϕ be defined as in the proof of Theorem 4.16. Let us set

$$\Lambda = \bigcup_{k=1}^{\infty} P_k, \quad \Lambda_{rd} = \{t \in [a,b) : t \in \Lambda \text{ and t is right-dense}\}. \tag{4.29}$$

$$G = \{t \in [a,b) : \quad f \text{ is discontinuous at } t\}, \quad G_{rd} = \{t \in G : \quad t \text{ is right-dense}\} \tag{4.30}$$

$$A = \{t \in [a,b) : \varphi(t) \neq \phi(t)\}. \tag{4.31}$$

Assume that t is a point in $[a,b)$ such that

$$\varphi(t) = f(t) = \phi(t) \tag{4.32}$$

and that t is not in Λ. Then f is continuous at t, otherwise there would exist $\varepsilon > 0$ and a sequence (t_j), $\lim t_j = t$, such that $|f(t_j) - f(t)| > \varepsilon$ for each j. But then $\phi(t) \geq \varphi(t) + \varepsilon$ which contradicts the assumption 4.32. Note that for each right-scattered point t of $[a,b)$, 4.32 holds. Indeed, it is easy to see that all right-scattered points of $[a,b)$ belong to Λ (that is, they are partition points). Therefore for each right-scattered point t of $[a,b)$ and all sufficiently large k, we have $\varphi_k(k) = \phi_k(t) = f(t)$ and hence $\varphi(t) = \phi(t)$. Consequently,

$$G_{rd} \subset (A \cup \Lambda_{rd}) \tag{4.33}$$

Further, $\mu_\Delta(A) = 0$ and $\mu(\Lambda_{rd}) = 0$ by the fact that Λ and therefore Λ_{rd}, is at most countable and that each right-dense point has Δ-measure zero so from 4.33 we get that $\mu_\Delta(G_{rd}) = 0$ and the first part of the theorem is proved. Conversely, suppose that the set of all right-dense points of $[a, b)$ at which f is discontinuous is of Δ-measure zero: $\mu_\Delta(G_{rd}) = 0$. For each positive integer k, we choose $\delta_k > 0$ ($\delta_k \to 0$ as $k \to \infty$) and a partition $P_k : a = t_0^{(k)} < t_1^{(k)} < ... < t_{n(k)}^{(k)} = b$ of $[a, b)$ such that $P_k \in \wp_{\delta_k}$ and that P_{k+1} is a refinement of P_k. Further, let φ_k, ϕ_k, φ, ϕ be defined as in the proof of Theorem 4.16 so for each k we have $\varphi_k \leq \varphi_{k+1}$, $\phi_k \geq \phi_{k+1}$, $\varphi_k \leq f \leq \phi_k$ and $\varphi(t) = \lim_{k\to\infty} \varphi_k$, $\phi(t) = \lim_{k\to\infty} \phi_k$, $\varphi(t) \leq f(t) \leq \phi(t)$, $t \in [a, b)$.

Now suppose that $t \in [a, b)$ is a right-dense point and that f is continuous at t. Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\sup f - \inf f < \varepsilon$, where the supremum and infimum are taken over $(t - \delta, t + \delta) = \{s \in [a, b) : t - \delta < s < t + \delta\}$. For all sufficiently large k, a subinterval of P_k containing t will be lain in $(t - \delta, t + \delta)$ and so $\phi_k(t) - \varphi_k(t) < \varepsilon$. But ε is arbitrary so $\varphi(t) = \phi(t)$. Next, at each right-scattered point t of $[a, b)$ we have, as above in the first part of the proof, $\varphi(t) = \phi(t)$.

So we have shown that $\varphi(t) = \phi(t)$ provided either t is right-dense and f is continuous at t or right scattered. Therefore, we have the inclusion $A \subset G_{rd}$ where A and G_{rd} defined by 4.30 and 4.31, respectively. Hence, since $\mu_\Delta(G_{rd}) = 0$ by our assumption, we get $\mu_\Delta(A) = 0$. Consequently, $\varphi(t) = \phi(t)$ holds Δ-a.e on $[a, b)$. This implies that $\int_{[a,b)} \varphi(s)\Delta s = \int_{[a,b)} \phi(s)\Delta s$ and therefore, we get $\lim_{k\to\infty} L(f, P_k) = \lim_{k\to\infty} U(f, P_k)$. This shows that f is Riemann Δ-integrable on $[a, b)$. □

Proposition 4.3 *Let $f : [a, b) \to \mathbb{R}$ be a monotone function. Then it is Lebesgue Δ-integrable.*

Proof Suppose first that f is an increasing function. It is convenient to show f is Riemann Δ-integrable so is Lebesgue Δ-integrable. We need to show that for any partition $P \in \wp_\delta$ as given in Definition 4.1,

$$U(f, P) - L(f, P) < \epsilon \text{ for every } \epsilon > 0.$$

where

$$U(f, P) = \sum_{t_i \in P} M_i(t_i - t_{i-1}) \text{ and } L(f, P) = \sum_{t_i \in P} m_i(t_i - t_{i-1})$$

with

$$M_i = \sup\{f(t) : t \in [t_{i-1}, t_i)\} \text{ and } m_i = \inf\{f(t) : t \in [t_{i-1}, t_i)\}$$

Thus,

$$U(f,P) - L(f,P) = \sum_{t_i \in P} (M_i - m_i)(t_i - t_{i-1}).$$

Now we need to consider two cases: where t_{i-1} is a right-dense point and where t_{i-1} is a right-scattered point for the partition $P \in \wp_\delta$, $\delta > 0$. For $t_i - t_{i-1} < \delta$, t_{i-1} is understood a right-dense point, for $t_i - t_{i-1} > \delta$, t_{i-1} is then a right-scattered point

$$\begin{aligned} \sum_{t_i \in P} (M_i - m_i)(t_i - t_{i-1}) &= \sum_{t_i \in (D_R \cap P)} (M_i - m_i)(t_i - t_{i-1}) \\ &+ \sum_{t_i \in (S_R \cap P)} (M_i - m_i)(t_i - t_{i-1}). \end{aligned}$$

i) For $t \in S_R \cap P$, then $M_i = m_i = f(t_{i-1})$. Thus

$$\sum_{t_i \in (S_R \cap P)} (M_i - m_i)(t_i - t_{i-1}) = \sum_{t_i \in (S_R \cap P)} (f(t_{i-1}) - f(t_{i-1}))(t_i - t_{i-1}) = 0$$

ii) For $t \in D_R \cap P$, then $M_i = f(t_i^-) = \lim_{t \to t_i} f(t)$ and $m_i = f(t_{i-1})$. Thus

$$\sum_{t_i \in D_R \cap P} (M_i - m_i)(t_i - t_{i-1}) = \sum_{t_i \in (D_R \cap P)} (f(t_i^-) - f(t_{i-1}))(t_i - t_{i-1}).$$

Whence

$$\begin{aligned} \sum_{t_i \in P} (M_i - m_i)(t_i - t_{i-1}) &= \sum_{t_i \in (D_R \cap P)} (f(t_i^-) - f(t_{i-1}))(t_i - t_{i-1}) \\ &\leq \sum_{t_i \in (D_R \cap P)} (f(t_i) - f(t_{i-1}))(t_i - t_{i-1}) \end{aligned}$$

where $t_i - t_{i-1} < \delta$ for any $\delta > 0$, so

$$\leq \sum_{t_i \in (D_R \cap P)} (f(t_i) - f(t_{i-1}))\delta$$

Taking δ as

$$\delta = \frac{\epsilon}{\sum_{t_i \in (D_R \cap P)} (f(t_i) - f(t_{i-1}))}$$

then we get

$$U(f,P) - L(f,P) < \epsilon$$

which implies Riemann Δ-integrability and Lebesgue Δ-integrability as well, this ends the proof. $\square$

CHAPTER 5

NOTES ON THE MEASURE THEORY ON TIME SCALE

In this chapter we will introduce Lebesgue-Stieltjes measure adapted to time scales and connect the new structure with the notion of classical Lebesgue-Stieltjes measure. We will begin with the concept of the Riemann-Stieltjes Δ-integral. The Dutch mathematician Thomas Stieltjes (1856-1894) studied on integrating one function with respect to another function. His ideas were originally developed as an extension of the Riemann integral, known as the Riemann-Stieltjes integral. The subsequent combination of his ideas with the measure theoretic approach of Lebesgue has resulted in a very powerful and flexible concept of integration. (Carter and Brunt 2000)

5.1. Riemann-Stieltjes-Δ Integral

Definition 5.1 *Let $\mathbb{T}$ be a time scale, $[a,b) \subset \mathbb{T}$ with $a \leq b$. A partition P of $[a,b)$ is a finite ordered set of points $P = \{t_0, t_1, ...t_n\}$ such that $a = t_0 < t_1 < ... < t_n = b$. A partition P' of $[a,b)$ is said to be finer than P if $P \subseteq P'$. Let $\alpha : \mathbb{T} \to \mathbb{R}$ is an increasing function. The symbol $\Delta\alpha_k(t)$ denotes the difference $\Delta\alpha_k(t) = \alpha(t_k)(t) - \alpha(t_{k-1})(t)$ so that*

$$\sum_{k=1}^{n} \Delta\alpha_k(t) = \alpha(b) - \alpha(a) \ \text{ where } \ \alpha : \mathbb{T} \to \mathbb{R}.$$

A sum of the form

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(\xi_k)\Delta\alpha_k(t)$$

is called a Riemann-Stieltjes Δ-sum of f with respect to α where $\xi_i \in [t_{k-1}, t_k)$.

We say that f is Riemann Δ-integrable with respect to α if for every $\epsilon > 0$, there exists $\delta > 0$ with $P \in \wp_\delta$ such that

$$|S(P, f, \alpha) - A| < \epsilon \ \text{ for every choice of points } \ \xi_k \in [t_{k-1}, t_k).$$

The unique number A is the Riemann Δ-integral of f from a to b with respect to α denoted by $\int_a^b f(s)\Delta\alpha(s)$. The functions f and α are referred to as the integrand and the *integrator respectively.*

Remark 5.1 *For the special case $\alpha(t) = t$, then the integral turns into Riemann Δ-integral and denoted by $\int_a^b f(s)\Delta s$.*

Remark 5.2 *Suppose that $\mathbb{T} = h\mathbb{Z}$, let $f : [a,b) \to \mathbb{R}$, be a bounded function, $\alpha : [a,b) \to \mathbb{R}$ be an increasing bounded function. Then the unique partition $P \in \wp_\delta$ is*

$$a = t_0 < t_1 = a + h < t_2 = a + 2h < ... < t_n = a + nh = b.$$

Then,

$$\begin{aligned}
\int_a^b f(s)\Delta\alpha(s) &= \int_a^{a+nh} f(s)\Delta\alpha(s) \\
&= \int_a^{a+h} f(s)\Delta\alpha(s) + \int_{a+h}^{a+2h} f(s)\Delta\alpha(s) + ... + \int_{a+(n-1)}^{a+nh} f(s)\Delta\alpha(s) \\
&= f(a)\big\{\alpha(a+h) - \alpha(a)\big\} + f(a+h)\big\{\alpha(a+2h) - \alpha(a+h)\big\} + ... \\
&\quad + f(a + (n-1)h)\big\{\alpha(a+nh) - \alpha(a + (n-1)h)\big\} \\
&= \sum_{k=1}^{n} f\big(a + (k-1)h\big)\big\{\alpha(a+kh) - \alpha\big(a + (k-1)h\big)\big\}.
\end{aligned}$$

Remark 5.3 *Suppose that $\alpha : [a,b) \to \mathbb{R}$, $\alpha(t) = c$, c is any constant. Then for any partition*

$$\int_a^b f(s)\Delta\alpha(s) = 0 \ \ \textit{since for any } k, \ \ \Delta\alpha_k(t) = 0.$$

Remark 5.4 *Suppose that $f : [a,b) \to \mathbb{R}$, $f(t) = c$, where c is any constant. Then for any partition*

$$\begin{aligned}
\int_a^b f(s)\Delta\alpha(s) &= c\int_a^b \Delta\alpha(s) \\
&= c\lim_{n\to\infty}\sum_{k=1}^{n}\big\{\alpha(t_k) - \alpha(t_{k-1})\big\} \\
&= c\lim_{n\to\infty}\big\{\alpha(t_1) - \alpha(t_0) + \alpha(t_2) - \alpha(t_1) + ... + \alpha(t_k) - \alpha(t_{k-1})\big\} \\
&= c\lim_{n\to\infty}\sum_{k=1}^{n}\big\{\alpha(t_k) - \alpha(t_0)\big\} \\
&= c\big(\alpha(b) - \alpha(a)\big).
\end{aligned}$$

5.2. Lebesgue-Stieltjes Δ-Measure

In this section, we define a new concept Lebesgue-Stieltjes Δ and ∇-measure and by using this measure, we will define an integral adapted to time scale specifically

Lebesgue-Stieltjes Δ-integral. We will also establish the relation between Lebesgue-Stieltjes measure and Lebesgue-Stieltjes Δ-measure and as a result, between Lebesgue-Stieltjes integral and Lebesgue-Stieltjes Δ- integral. The original Lebesgue-Stieltjes measure is defined by introducing a premeasure μ on all intervals of $\mathbb{R}$ as

i) $\mu([a,b)) = \alpha(b^-) - \alpha(a^-)$,

ii) $\mu([a,b]) = \alpha(b^+) - \alpha(a^-)$,

iii) $\mu((a,b]) = \alpha(b^+) - \alpha(a^+)$,

iv) $b > a$, $\mu((a,b)) = \alpha(b^-) - \alpha(a^+)$,

where $\alpha : \mathbb{R} \to \mathbb{R}$ is an increasing function and

$$\alpha(a^-) = \lim_{t \to a^-} \alpha(t) \text{ and } \alpha(a^+) = \lim_{t \to a^+} \alpha(t).$$

Our aim is to generalize this measure to time scales. We will begin with defining a premeasure $m_1^\alpha : \mathbb{T} \to [0, +\infty]$ on $\Im^\alpha$, the family of all intervals of $\mathbb{T}$ by using a monotone increasing function $\alpha : \mathbb{T} \to \mathbb{R}$, taking domain into account, as

i) $m_1^\alpha([a,b)) = \alpha(b^-) - \alpha(a^-)$,

ii) $m_1^\alpha([a,b]) = \alpha(\sigma(b)^+) - \alpha(a^-)$,

iii) $m_1^\alpha((a,b]) = \alpha(\sigma(b)^+) - \alpha(\sigma(a)^+)$,

and if $b > \sigma(a)$,

iv) $m_1^\alpha((a,b)) = \alpha(b^-) - \alpha(\sigma(a)^+)$.

The open interval $(a, \sigma(a))$ is empty set and we define $m_1^\alpha(a, \sigma(a))$ to be zero. Obviously, $[a,a)$ and $(a,a]$ are also empty sets and have premeasures zero but it is seen from the definition and need not be specified separately. It is obvious that m_1^α is a countably additive measure on $\Im^\alpha$.

The Lebesgue-Stieltjes Δ-outer measure $(m_1^\alpha)^*$ associated with α is the function defined on all subsets of $\mathbb{T}$ by

$$(m_1^\alpha)^*(E) = \inf \sum_{i=1}^{\infty} m_1^\alpha(I_n)$$

provided that there exists at least one finite or countable covering system of intervals $I_n \subset \Im^\alpha$ of E as $E \subset \bigcup_{n=1}^{\infty} I_n$. If there is no such covering of E we say $(m_1^\alpha)^*(E) = \infty$. Let $A \subset \mathbb{T}$. If

$$(m_1^\alpha)^*(A) = (m_1^\alpha)^*(A \cap E) + (m_1^\alpha)^*(A \cap E^c)$$

holds, then we say E is $(m_1^\alpha)^*$-measurable (or α_Δ). $\mathbf{M}((m_1^\alpha)^*)$, the family of all $(m_1^\alpha)^*$-measurable subset of $\mathbb{T}$ forms a σ-algebra. We restrict $(m_1^\alpha)^*$ to $\mathbf{M}((m_1^\alpha)^*)$ and denote by μ_Δ^α. This is the Lebesgue-Stieltjes Δ-measure generated by α.

Remark 5.5 *All intervals on $\mathbb{T}$ are α_Δ-measurable since any interval can be covered by itself which is the smallest cover, thus for any interval I, premeasure $m_1^\alpha(I)$ and α_Δ-measure $\mu_\Delta^\alpha(I)$ coincide.*

i) $\mu_\Delta^\alpha([a,b)) = \alpha(b^-) - \alpha(a^-)$,

ii) $\mu_\Delta^\alpha([a,b]) = \alpha(\sigma(b)^+) - \alpha(a^-)$,

iii) $\mu_\Delta^\alpha((a,b]) = \alpha(\sigma(b)^+) - \alpha(\sigma(a)^+)$,

iv) $b > \sigma(a)$, $\mu_\Delta^\alpha((a,b)) = \alpha(b^-) - \alpha(\sigma(a)^+)$.

Proposition 5.1 *Let $\{c\} \subset \mathbb{T}$. Then it is μ_Δ^α-measurable and*

$$\mu_\Delta^\alpha(\{c\}) = \mu_\Delta^\alpha([c,c]) = \alpha(\sigma(c)^+) - \alpha(c^-). \tag{5.1}$$

Proof It is obvious from Remark 5.5 and single point set is covered by itself as closed interval, which is the smallest cover. $\square$

Any choice of interval while evaluating the α-measure of a single point set gives us different results. Although $[c,c]$ and $[c,\sigma(c))$ has the same Δ-measure, it differs while considering α_Δ-measure, because in α_Δ-measure, we concern about one sided limits of an increasing function α, at the endpoints of given intervals. Furthermore, theory does not contradict. Considering these two intervals: [c,c] and $[c,\sigma(c))$, corresponding α_Δ-measures:

$$\begin{aligned} \mu_\Delta^\alpha([c,c]) &= \alpha(\sigma(c)^+) - \alpha(c^-). &(5.2)\\ \mu_\Delta^\alpha([c,\sigma(c))) &= \alpha(\sigma(c)^-) - \alpha(c^-). &(5.3)\end{aligned}$$

Comparing the right hand side of the equalities: $\mu_{\Delta}^{\alpha}([c,c]) \geq \mu_{\Delta}^{\alpha}([c,\sigma(c)))$. Taking into monotone property of measure, the reason is

$$[c,c] \supseteq [c,\sigma(c)). \tag{5.4}$$

In order to show Inequality 5.4, we need to consider two cases: where c is a right-scattered and where c is a right-dense point of interior of $\mathbb{T}$. Let c is a right-scattered point, then $\sigma(t) > t$, thus $[c,c] = [c,\sigma(c))$. Let c is a right-dense point, then $\sigma(t) = t$, thus $[c,\sigma(c))$ corresponds the empty set which is included by any other set, thus $[c,c] \supseteq [c,\sigma(c))$. Summarizing these two results, we obtain $[c,c] \supseteq [c,\sigma(c))$, consequently $\mu_{\Delta}^{\alpha}([c,c]) \geq \mu_{\Delta}^{\alpha}([c,\sigma(c)))$.

Remark 5.6 *From Proposition 5.1, is it seen why Δ-measure of $\max \mathbb{T}$ is infinity. The reason is that we are not able to approach to $\max \mathbb{T}$ from the right hand side. Furthermore we are not allowed to take the limit of the minimum point of $\mathbb{T}$ from the left hand side. Thus, we say α_{Δ}-measures of maximum and minimum points of a time scale and also α_{Δ}-measure of any set containing at least one of them are undefined (∞).*

However, up to now, Δ-measure of the minimum point of a bounded below time scale has been introduced as an ordinary interior point of the time scale because the formula $\mu_{\Delta}(\{t_0\}) = \sigma(t_0) - t_0$ stays the same but extending theory, replacing measure with respect to an increasing function, it is seen that a single point set has α_{Δ}-measure $\alpha(\sigma(t_0)^+) - \alpha(t_0^-)$ and because of the fact that the limit from left hand side at t_0 is undefined, we finally get this result.

Let $\mathbb{T} = \mathbb{R}$, then α_{Δ} and α measures coincide since for all $t \in \mathbb{T}$, $\sigma(t) = t$. Let $\mathbb{T} = \mathbb{Z}$, then

i) $\mu_{\Delta}^{\alpha}([a,b)) = \alpha(b) - \alpha(a)$,

ii) $\mu_{\Delta}^{\alpha}([a,b]) = \alpha(b+1) - \alpha(a)$,

iii) $\mu_{\Delta}^{\alpha}((a,b]) = \alpha(b+1) - \alpha(a+1)$,

iv) For $b > a+1$, $\mu_{\Delta}^{\alpha}((a,b)) = \alpha(b) - \alpha(a+1)$.

Here we are not interested in right-sided and left-sided limits because all functions defined on $\mathbb{Z}$ is continuous by Proposition 3.4.

Let $\alpha : \mathbb{T} \to \mathbb{T}$, $\alpha(t) = t$, then α_Δ-measure turns in to Δ-measure introduced by Guseinov. (Guseinov 2003)

i) $\mu_\Delta^\alpha([a,b)) = b - a$,

ii) $\mu_\Delta^\alpha([a,b]) = \sigma(b) - a$,

iii) $\mu_\Delta^\alpha((a,b]) = \sigma(b) - \sigma(a)$,

iv) For $b > \sigma(a)$, $\mu_\Delta^\alpha((a,b)) = b - \sigma(a)$.

Let us introduce the concept of Stieltjes ∇-measure on time scales. Let $\alpha : \mathbb{T} \to \mathbb{R}$ be a monotone increasing function. We define a set function m_2^α on the family of all intervals of $\mathbb{T}$ denoted by $\Im^\alpha$ as follows:

i) $m_2^\alpha([a,b)) = \alpha(\rho(b)^-) - \alpha(\rho(a)^-)$,

ii) $m_1^\alpha([a,b]) = \alpha(b^+) - \alpha(\rho(a)^-)$,

iii) $m_1^\alpha((a,b]) = \alpha(b^+) - \alpha(a^+)$,

For $a < \rho(b)$,

iv) $m_1^\alpha((a,b)) = \alpha(\rho(b)^-) - \alpha(a^+)$,

where The Lebesgue-Stieltjes ∇-outer measure $(m_2^\alpha)^*$ of a set E associated with α is the function defined on all subsets of $\mathbb{T}$ by $(m_2^\alpha)^*(E) = \inf \sum_{i=1}^{\infty} (m_2^\alpha)(I_n)$ provided that there exists at least one finite or countable covering of intervals $I_n \subset \Im^\alpha$ of E such that $E \subset \bigcup_{n=1}^{\infty} I_n$. If there is no such covering of E, we say that $(m_2^\alpha)^*(E) = \infty$.

By restriction of the outer measure to the family of all α_∇-measurable sets we obtain a countably additive measure denoted by μ_∇^α. Similarly, any set includes maximum or minimum of a time scale has α_∇-measure infinity.

Remark 5.7 *All intervals on $\mathbb{T}$ are α_∇-measurable since any interval can be covered by itself which is the smallest cover, thus for any interval I, premeasure $m_2^\alpha(I)$ and α_∇-measure $\mu_\nabla^\alpha(I)$ coincide so*

i) $\mu_\nabla^\alpha([a,b)) = \alpha(\rho(b)^-) - \alpha(\rho(a)^-)$,

ii) $\mu_\nabla^\alpha([a,b]) = \alpha(b^+) - \alpha(\rho(a)^-)$,

iii) $\mu_{\nabla}^{\alpha}((a,b]) = \alpha(b^+) - \alpha(a^+)$,

For $a < \rho(b)$,

iv) $\mu_{\nabla}^{\alpha}((a,b)) = \alpha(\rho(b)^-) - \alpha(a^+)$.

Let $\mathbb{T} = \mathbb{R}$, then α_{∇}-measure and α-measure coincide since for all $t \in \mathbb{T}$, $\rho(t) = t$. Let $\mathbb{T} = \mathbb{Z}$, then,

i) $\mu_{\nabla}^{\alpha}([a,b)) = \alpha(b-1) - \alpha(a-1)$,

ii) $\mu_{\nabla}^{\alpha}([a,b]) = \alpha(b) - \alpha(a-1)$,

iii) $\mu_{\nabla}^{\alpha}((a,b]) = \alpha(b) - \alpha(a)$,

iv) For $b > a+1$, $\mu_{\nabla}^{\alpha}((a,b)) = \alpha(b-1) - \alpha(a)$.

Let $\alpha : \mathbb{T} \to \mathbb{T}$, $\alpha(t) = t$, then α_{Δ}-measure turns in to Δ-measure introduced by Guseinov. (Guseinov 2003)

i) $\mu_{\nabla}^{\alpha}([a,b)) = \rho(b) - \rho(a)$,

ii) $\mu_{\nabla}^{\alpha}([a,b]) = b - \rho(a)$,

iii) $\mu_{\nabla}^{\alpha}((a,b]) = b - a$,

iv) For $b > \sigma(a)$, $\mu_{\nabla}^{\alpha}((a,b)) = \rho(b) - a$.

Proposition 5.2 *Let* $\{c\} \subset \mathbb{T}$. *Then it is* μ_{∇}^{α}*-measurable and*

$$\mu_{\nabla}^{\alpha}(\{c\}) = \mu_{\nabla}^{\alpha}([c,c]) = \alpha(c^+) - \alpha(\rho(c)^-). \tag{5.5}$$

Proof Proof is obvious from Remark 5.7.

Example 5.8 *Let* $\mathbb{T} = [0,3] \cup \{4\} \cup [6,9]$ *and*

$$\alpha(t) = \begin{cases} 3 - e^{-t} & if\ 0 \leq t \leq 1 \\ 4 & if\ 1 < t < 3 \\ 2t+1 & if\ 3 \leq t < 7 \\ t^2 & if\ 7 \leq t \leq 9 \end{cases}$$

Find the α_{Δ}*-measure and* α_{∇}*-measure of the following sets:*
$\{4\},\ [3,6),\ (8,9],\ \{3\},\ \{7\},\ [0,1)$.

Solution. Let us first consider the α_Δ-measures of the sets.

a) $\mu_\Delta^\alpha(\{4\}) = \mu_\Delta^\alpha([4,4]) = \alpha(\sigma(4)^+) - \alpha(4^-) = \alpha(6^+) - \alpha(4^-) = 4.$

b) $\mu_\Delta^\alpha([3,6)) = \alpha(6^-) - \alpha(3^-) = 9.$

c) $\mu_\Delta^\alpha((8,9]) = \alpha(\sigma(9)^+) - \alpha(\sigma(8)^-) = \alpha(9^+) - \alpha(8^-) = \infty$ since $\alpha(9^+)$ is not defined.

d) $\mu_\Delta^\alpha(\{3\}) = \mu_\Delta^\alpha([3,3]) = \alpha(\sigma(3)^+) - \alpha(3^-) = \alpha(4^+) - \alpha(3^+) = 3.$

e) $\mu_\Delta^\alpha([7,8]) = \alpha(\sigma(8)^+) - \alpha(7^-) = \alpha(8^+) - \alpha(7^-) = 49.$

f) $\mu_\Delta^\alpha([0,1)) = \alpha(1^-) - \alpha(0^-) = \infty$ since $\alpha(0^-)$ is not defined.

Now, let us consider α_∇-measures of the given sets.

a) $\mu_\nabla^\alpha(\{4\}) = \mu_\nabla^\alpha([4,4]) = \alpha(4^+) - \alpha(\rho(4)^-) = \alpha(4^+) - \alpha(3^-) = 5.$

b) $\mu_\nabla^\alpha([3,6)) = \alpha(\rho(6)^-) - \alpha(\rho(3)^-) = \alpha(4^-) - \alpha(3^-) = 5.$

c) $\mu_\nabla^\alpha((8,9]) = \alpha(9^+) - \alpha(8^+) = \infty$ since $\alpha(9^+)$ is not defined.

d) $\mu_\nabla^\alpha(\{3\}) = \mu_\nabla^\alpha([3,3]) = \alpha(3^+) - \alpha(\rho(3)^-) = \alpha(3^+) - \alpha(3^-) = 3.$

e) $\mu_\nabla^\alpha([7,8]) = \alpha(8^+) - \alpha(\rho(7)^-) == \alpha(8^+) - \alpha(7^-) = 39.$

f) $\mu_\nabla^\alpha([0,1)) = \alpha(\rho(1)^-) - \alpha(\rho(0)^-) = \infty$ since $\alpha(\rho(0)^-)$ is not defined.

5.3. Lebesgue-Stieltjes Measurable Sets and Lebesgue-Stieltjes Δ-Measurable Sets

In order to compare Lebesgue-Stieltjes measurable sets and Lebesgue-Stieltjes Δ-measurable sets, we need to extend α because α-measure of intervals $(t_i, \sigma(t_i))$ for t_i is any right-scattered point is not defined. Obviously, $\alpha(\sigma(t_i)^-)$ which is defined on $\mathbb{T}$ and equals to $\alpha(\sigma(t_i))$ since any function is left continuous at left-scattered point, also $\alpha(t_i^+)$ is defined on $\mathbb{T}$ and equals to $\alpha(t_i)$ since any function is right continuous at right-scattered points, the interval seems to have α-measure $\alpha(\sigma(t_i)) - \alpha(t_i)$. Although it is practically correct, the approach is theoretically wrong because of the fact that for α-measure of $(t_i, \sigma(t_i))$, α has to be defined on a set that includes this interval. Thus, we need to extend

α. The extension could be any function whose reduction to $\mathbb{T}$ corresponds α, monotone everywhere and continuous at scattered points. A proper choice would be as follows:

$$\alpha^{\sim}(t) = \begin{cases} \alpha(t) & if\ t \in \mathbb{T} \\ \dfrac{\alpha(\sigma(t_i)) - \alpha(t_i)}{\sigma(t_i) - t_i} t & if\ t \in (t_i, \sigma(t_i)) \end{cases} \tag{5.6}$$

which is a linear function at each $(t_i, \sigma(t_i))$, where t_is are right-scattered points and continuous not only at right-scattered, but also at left-scattered points provided that we are able to write $\alpha^{\sim}(t_i^+) = \alpha(t_i)$ where t_i is a right-scattered point and $\alpha^{\sim}(t_i^-) = \alpha(t_i)$ where t_i is a left-scattered point.

Remark 5.9 *When we consider α-measure, in order to extend α we will follow this procedure, but for any other functions extension, the previous method (see Equation 3.4) stays the same.*

Remark 5.10 *Although we know the usual Lebesgue measure of any measurable subsets of intervals of the form $(t_i, \sigma(t_i))$, we are not able to evaluate Lebesgue-Stieltjes Δ-measure of such subsets since even any point's α-measure depends on a function that we generate. Thus, we can just find the relation between α-measure and α_Δ-measure of a subset of $\mathbb{T}$.*

Proposition 5.3 *Let $[a, b)$ be a half closed bounded interval of $\mathbb{T}$ with $a, b \in \mathbb{T} - \{\min \mathbb{T}\}$. Then*

i) $\mu_\Delta^\alpha([a, b)) = \mu^{\alpha^{\sim}}([a, b)) + \sum_{i \in I_{[a,b)}} (\alpha(\sigma(t_i)) - \alpha(t_i))$.

ii) $\mu_\Delta^\alpha([a, b)) = \mu^{\alpha^{\sim}}([a, b)^{\sim})$.

where $[a, b)^{\sim}$ is the extension of set as given in Remark 2.4 and $\mu^{\alpha^{\sim}}$ is the Lebesgue-Stieltjes measure generated by $\alpha^{\sim}$.

Proof

i) Suppose that $\{t_1, t_2, ...t, b\} \subset [a, b) \cap S_R = I_{[a,b)}$ such that $a \leq t_1 \leq t_2 \leq ... \leq b$. Let t be the largest element of the sequence of right-scattered points. Then $[a, b)$ can be written as follows:

$$\begin{aligned}
[a,b) &= [a,t_1] \cup [\sigma(t_1),t_2] \cup ... \cup [t,b), \quad \text{so} \\
\mu^{\alpha^\sim}([a,b)) &= \mu^{\alpha^\sim}([a,t_1] \cup [\sigma(t_1),t_2] \cup ... \cup [t,b)) \\
&= \mu^{\alpha^\sim}\big([a,t_1]\big) + \mu^{\alpha^\sim}\big([\sigma(t_1),t_2]\big) + ... + \mu^{\alpha^\sim}\big([t,b)\big) \\
&= \alpha^\sim(t_1^+) - \alpha^\sim(a^-) + \alpha^\sim(t_2^+) - \alpha^\sim(\sigma(t_1)^-) + ... + \alpha^\sim(b^-) - \alpha^\sim(\sigma(t)^-) \\
&= \alpha^\sim(b^-) - \alpha^\sim(a^-) - \sum_{i\in I_{[a,b)}} (\alpha^\sim(\sigma(t_i)^-) - \alpha^\sim(t_i^+)) \\
&= \alpha(b^-) - \alpha(a^-) - \sum_{i\in I_{[a,b)}} (\alpha(\sigma(t_i)) - \alpha(t_i)).
\end{aligned}$$

Thus, we obtain

$$\mu^{\alpha^\sim}([a,b)) = \mu_\Delta^\alpha([a,b)) - \sum_{i\in I_{[a,b)}} (\alpha(\sigma(t_i)) - \alpha(t_i)).$$

ii) $(\alpha(\sigma(t_i)) - \alpha(t_i)) = (\alpha(\sigma(t_i)^-) - \alpha(t_i)^+) = \mu^{\alpha^\sim}((t_i,\sigma(t_i)))$,
so $\mu^{\alpha^\sim}([a,b)) - \sum_{i\in I_{[a,b)}} (\alpha(\sigma(t_i)) - \alpha(t_i)) = \mu^{\alpha^\sim}([a,b)^\sim)$, and we get the desired result. $\square$

Remark 5.11 *Obviously we can generalize Proposition 5.3 to any α_Δ-measurable set $E \subset \mathbb{T} - \{\max \mathbb{T}, \min \mathbb{T}\}$ as*

i) $\mu_\Delta^\alpha(E) = \mu^{\alpha^\sim}(E) + \sum_{i\in I_E}(\alpha(\sigma(t_i)) - \alpha(t_i))$.

ii) $\mu_\Delta^\alpha(E) = \mu^{\alpha^\sim}(E^\sim)$.

5.4. Lebesgue-Stieltjes Δ-Integral

We will begin with considering Lebesgue-Stieltjes Δ-integral of a simple function.

Definition 5.2 *Let $\mathbb{T}$ be a time scale, $\alpha : \mathbb{T} \to \mathbb{R}$ be an increasing function, μ_Δ^α be the Lebesgue-Stieltjes Δ-measure defined on $\mathbb{T}$, $S : \mathbb{T} \to \mathbb{R}$ be a nonnegative α_Δ-measurable simple function such that $S(t) = \sum_{i=1}^{n} a_i \chi_{A_i}$ where A_is are pairwise disjoint α_Δ-measurable sets with $A_i = \{t : S(t) = a_i\}$. Then we define Lebesgue-Stieltjes Δ-integral of S on a α_Δ-measurable set E as*

$$\int_E S(s)\Delta\alpha(s) = \sum_{i=1}^{n} a_i \mu_\Delta^\alpha(A_i \cap E). \tag{5.7}$$

If for some k, $a_k = 0$ and $\mu_\Delta^\alpha(A_k \cap E) = \infty$, we define $a_k \mu_\Delta^\alpha(A_k \cap E) = \infty$.

Example 5.12 *Let α and $\mathbb{T}$ be defined in Example 5.8. Let*

$$S_1(t) = \begin{cases} 1 & if\ 0 \le t \le 3 \\ 4 & if\ 3 < t \le 9 \end{cases}$$

and let

$$S_2(t) = \begin{cases} 1 & if\ 0 \le t < 3 \\ 4 & if\ 3 \le t \le 9. \end{cases}$$

Evaluate the integral of S_1 and S_2 on $[1, 8]$ with respect to α and compare the results.

Solution. $[0, 3] \cap [1, 8] = [1, 3]$ and $(3, 9] \cap [1, 8] = (3, 8]$ and α_Δ-integral of S_1 on $[1, 8]$ is

$$\int_{[1,8]} S_1(s)\Delta\alpha(s) = 1.\mu_\Delta^\alpha([1, 3]) + 4.\mu_\Delta^\alpha((3, 8])$$

where

$$\begin{aligned} \mu_\Delta^\alpha([1, 3]) &= \alpha(\sigma(3)^+) - \alpha(1^-) \\ &= \alpha(4^+) - \alpha(1^-) \\ &= 9 - (3 - e^{-1}) \\ &= 6 + e^{-1} \end{aligned}$$

and

$$\begin{aligned} \mu_\Delta^\alpha((3, 8]) &= \alpha(\sigma(8)^+) - \alpha(\sigma(3)^+) \\ &= \alpha(8^+) - \alpha(\sigma(3)^+) \\ &= 64 - 9 \\ &= 55. \end{aligned}$$

Thus, we have

$$\begin{aligned} \int_{[1,8]} S_1(s)\Delta\alpha(s) &= 1.(6 + e^{-1}) + 4.55 \\ &= 226 + e^{-1}. \end{aligned}$$

$[0,3) \cap [1,8] = [1,3)$ and $[3,9] \cap [1,8] = [3,8]$ and from the definition,
the α_Δ-integral of S_2 on $[1,8]$ is

$$\int_{[1,8]} S_2(s)\Delta\alpha(s) = 1.\mu_\Delta^\alpha([1,3)) + 4.\mu_\Delta^\alpha([3,8]).$$

where

$$\begin{aligned}\mu_\Delta^\alpha([1,3)) &= \alpha(3^-) - \alpha(1^-) \\ &= 4 - (3 - e^{-1}) \\ &= 1 + e^{-1}\end{aligned}$$

and

$$\begin{aligned}\mu_\Delta^\alpha([3,8]) &= \alpha(\sigma(8)^+) - \alpha(3^-) \\ &= \alpha(8^+) - \alpha(3^-) \\ &= 64 - 4 \\ &= 60.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{[1,8]} S_1(s)\Delta\alpha(s) &= 1.(1 + e^{-1}) + 4.60 \\ &= 241 + e^{-1}.\end{aligned}$$

Although these two simple functions are nearly the same, the reason of the difference of integrals is the change of the discontinuity point.

Lemma 5.13 *Let E be an α_Δ-measurable set of $\mathbb{T} - \{\max\mathbb{T}, , \min\mathbb{T}\}$, let $S : \mathbb{T} \to \mathbb{R}$ be a simple function with $S(t) = \sum_{i=1}^{n} a_i \chi_{A_i}$ where A_is are pairwise disjoint α_Δ-measurable sets with $A_i = \{t : S(t) = a_i\}$ and $S^\sim = \sum_{i=1}^{n} a_i \chi_{A_i^\sim}$ be the extension of S as Equation 3.4 and $A_i^\sim$ and $E_i^\sim$ be the extensions of A_i and E_i respectively as in Remark 2.4, $\alpha^\sim$ be the extension of α as in Equation 5.6 and corresponding measures be denoted by $\mu^{\alpha^\sim}$ and μ_Δ^α usual Lebesgue-Stieltjes measure and Lebesgue-Stieltjes Δ-measure respectively. Then*

$$\int_E S(s)\Delta\alpha(s) = \int_{E^\sim} S^\sim(s) d\alpha^\sim(s).$$

Proof We will use the fact that for each c_i, $S(t) = c_i$ for $t \in A_i$, then $S^{\sim}(t) = c_i$ for $t \in A_i^{\sim}$. Furthermore, $\mu_{\Delta}^{\alpha}(A_i \cap E) = \mu^{\alpha^{\sim}}(A_i^{\sim} \cap E^{\sim}) = \mu^{\alpha^{\sim}}(A_i \cap E)^{\sim}$. Multiplying by a_i and summing from $i = 1$ to $i = n$ we have

$$\sum_{i=1}^{n} a_i \mu_{\Delta}^{\alpha}(A_i \cap E) = \sum_{i=1}^{n} a_i \mu^{\alpha^{\sim}}(A_i \cap E)^{\sim}.$$

We get the integral of $S(t)$ on measurable set E with respect to α on the left hand side of the equation and the integral of $S^{\sim}(t)$ on measurable set $E^{\sim}$ with respect to $\alpha^{\sim}$ on the right hand side of the equation. Thus we get

$$\int_E S(s)\Delta\alpha(s) = \int_{E^{\sim}} S^{\sim}(s) d\alpha^{\sim}(s).$$

We know that if a set $A \subset \mathbb{T}$ is α_{Δ}-measurable, then A is α-measurable. $\square$

Definition 5.3 *Let $E \subset \mathbb{T}$ be an α_{Δ}-measurable set and let $f : \mathbb{T} \to [0, +\infty]$ be an α_{Δ}-measurable function. The Lebesgue-Stieltjes Δ-integral of f on E is defined as*

$$\int_E f(s)\Delta\alpha(s) = \sup \int_E S(s)\Delta\alpha(s)$$

where the supremum is taken over all α_{Δ}-measurable nonnegative simple functions S such that $S \leq f$ defined on $\mathbb{T}$.

The next lemma exhibits the Lebesgue-Stieltjes integral and Lebesgue-Stieltjes Δ-integral of functions. Since the proof is similar to comparison of Lebesgue Δ-integral and usual Lebesgue integral we do not give the proof.

Lemma 5.14 *Let $E \subset \mathbb{T} - \{\max \mathbb{T}, \min \mathbb{T}\}$ be an α_{Δ}-measurable set, $f : \mathbb{T} \to [0, +\infty]$ be an α_{Δ}-measurable function and $f^{\sim}$ be the extension of f as in Equation* 3.4. *Then*

$$\int_E f(s)\Delta\alpha(s) = \int_{E^{\sim}} f^{\sim}(s) d\alpha(s)$$

where $E^{\sim}$ denotes the set defined in Remark 2.4.

Definition 5.4 *Let $E \subset \mathbb{T}$ be a α_{Δ}-measurable set and let $f : \mathbb{T} \to \overline{\mathbb{R}}$ be a α_{Δ}-measurable function. We say that f is Lebesgue-Stieltjes Δ-integrable on E if at least one of the elements $\int_E f^{+}(s)\Delta\alpha(s)$ or $\int_E f^{-}(s)\Delta\alpha(s)$ is finite, where the positive and negative parts of f, f^{+} and f^{-} respectively. In this case, we define the Lebesgue-Stieltjes Δ-integral of f on E as*

$$\int_E f(s)\Delta\alpha(s) = \int_E f^{+}(s)\Delta\alpha(s) - \int_E f^{-}(s)\Delta\alpha(s).$$

Theorem 5.15 *Let $E \subset \mathbb{T} - \{\max \mathbb{T}, \min \mathbb{T}\}$ be an α_Δ-measurable set. If $f : \mathbb{T} \to \overline{\mathbb{R}}$ is α_Δ-integrable on E, then*

$$\int_E f(s)\Delta\alpha(s) = \int_E f(s)d\alpha(s) + \sum_{i \in I_E} f(t_i)(\alpha(\sigma(t_i)) - \alpha(t_i))$$

with I_E denotes the set of indices of the right-scattered points of E.

Proof

$$\begin{aligned}
\int_{E^\sim} f^\sim(s)d\alpha^\sim(s) &= \int_{E \cup \bigcup (t_i, \sigma(t_i))} f^\sim(s)d\alpha^\sim(s) \\
&= \int_E f^\sim(s)d\alpha^\sim(s) + \int_{\bigcup (t_i, \sigma(t_i))} f^\sim(s)d\alpha^\sim(s) \\
&= \int_E f(s)d\alpha(s) + \sum_{i \in I_E} \int_{(t_i, \sigma(t_i))} f(t_i)d\alpha^\sim(s) \\
&= \int_E f(s)d\alpha(s) + \sum_{i \in I_E} f(t_i)(\alpha(\sigma(t_i)) - \alpha(t_i)).
\end{aligned}$$

and we conclude that by Lemma 5.14,

$$\int_E f(s)\Delta\alpha(s) = \int_E f(s)d\alpha(s) + \sum_{i \in I_E} f(t_i)(\alpha(\sigma(t_i)) - \alpha(t_i)). \qquad \square$$

CHAPTER 6

CONCLUSION

The goal of this study is to adapt basic concepts of measure theory to time scales. For this purpose, we began with expanding three basic papers on this subject. We have tried to make the idea of the related papers more general. We have made a specific study on Δ-measurable functions which did not take place on these papers.

These works inspired us to adapt another concept known as Stieltjes measure to time scales. In these thesis, we have improved a new concept Stieltjes measure on time scales, which is general form of measure on time scales. Lebesgue-Stieltjes measure is a general form of classical Lebesgue measure so Stieltjes measure adapted to time scales had to be a general form of measure on time scales. Thus, we have constructed a new measure, depending on an increasing function α, which coincides with the classical Lebesgue-Stieltjes measure, taking time scale as real numbers and which coincides with Lebesgue measure on time scales, which was introduced by Guseinov, taking α as the identity function defined on the same time scale.

It was known that the maximum point of a time scale, or any Δ-measurable set containing this maximum point is infinity. Generalizing the measure, we have found out that, not only the measure of the maximum, but also the measure of the minimum point of a time scale or the measure of any Δ-measurable set containing this minimum point is undefined which was not seen on Lebesgue measure on time scales.

With the help of this new structure, we have constructed an integral, Lebesgue-Stieltjes integral on time scales. Moreover, we have established the relationship between these two integrals, classical Lebesgue-Stieltjes integral and the Lebesgue-Stieltjes integral by using similar idea of Cabada and Vivero.

REFERENCES

Aliprantis, C.D. and Burkinshaw, O., 1998. *"Principles of Real Analysis"*, (Academic Press, San Diego).

Bartle, R. G., 1992. *"Introduction to Real Analysis"*, (John Wiley and Sons, New York).

Bohner, M. and Peterson, A., 2001. *"Dynamic Equations on Time Scales;an Introduction with Applications"*, (Birkhauser, Boston).

Cabada, A. and Vivero, D. 2004. *"Expression of the Lebesgue Δ-Integral on Time Scales as a Usual Lebesgue Intregral; Application to the Calculus of Δ-Antiderivatives"*,Elseiver 4, pp.291-310.

Carter M. and Brunt B. V., 2000. *"The Lebesgue-Stieltjes Integral"*, (Springer-Verlag, NewYork).

Cohn, D., 1997. *"Measure Theory"*, (Birkhuser, Boston).

Craven, B. D., 1982. *"Lebesgue Measure and Integral"*, (Pitmann Publishing, Edinburgh).

Dönmez, A., 2001 *"Reel Analiz, Lebesgue Ölçümü ve İntegrali"*, (Seçkin Yayıncılık, Ankara).

Guseinov S. G. and Kaymakcalan B., 2002. *"Basics on Riemann Delta and Nabla Integration on Time Scales"*, J. Difference Equations Appl., No. 8, pp. 1001-1017.

Guseinov, S.G., 2003. *"Integration on Time Scales"*, Elseiver Academic Press, No. 285, pp. 107-127.

Guseinov S. G. and Bohner M., 2003. *"Riemann and Lebesgue Integration"*, Advances in Dynamic Equations on Time Scales, pp. 117-163.

Roussas, G. G., 2004. *"An Introduction to Measure Theoretic Probability"*, (Elseiver Academic Press, California).

Royden, H.L., 1988. *"Real Analysis"*, (Macmillan, New York).

Rudin, W., 1987. *"Real and Complex Analysis"*, (McGraw-Hill, New York).

Rzezuchowski, T., 2005. *"A Note on Measures on Time Scales"*,Demonstratio Mathematica, Vol.38, No. 1 pp. 79-84.